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खण्ड Volume 5 - 8 (1991 - 1994)

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

JOURNAL OF NATURAL & PHYSICAL SCIENCES

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Journal of Natural & Physical Sciences

शोध पत्रिका पटल

अध्यक्ष – धर्मपाल आर्य, कुलपति

उपाध्यक्ष – एस० एल० सिंह, डीन

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 1-10

'CHEMICAL INVESTIGATIONS OF SOME INDIGENOUS BASIL OILS'

Satinder Kaur* & Gopal K. Sinha*

(Received 30-01-1991)

ABSTRACT

Chemical composition of essential oil of *Ocimum sanctum* Linn distributed in and around Rishikesh and from the plants raised at Haldwani from the seeds of the plants distributed at Rishikesh, have been studied by derivatisation, TLC, GLC, PMR, MS and IR. In all twenty compounds have been identified as against nine compounds reported in literature, Principal components are : eugenol, methyl eugenol, and methyl chavicol. The Rishikesh oil sample containing 66.5 percent eugenol can serve as a substitute for oil of cloves.

Keywords and Phrases : Essential oil, *Ocimum sanctum*, Chemistry-subject classification:Pharmaceutical analysis Essential Oils - 62.

INTRODUCTION

Ocimum sanctum, belonging to N.O. Labiateae, is widely distributed in the warmer parts of the world. In India it is commonly known as 'Sri tulsi' and 'Krishna tulsi' and holds promise due to its high eugenol content. It is held sacred and the plant and its parts are credited possessing numerous curative effects probably not shared by many other plants.

Essential oil from *O. sanctum* has not been thoroughly investigated. Only a few constituents have been reported viz. linalool, geraniol, cirtronellol, cineole and methyl chavicol in the oil of plants raised at Phillipines by Guenther [5]; Cineole, eugenol and methyl chavicol by Gildmeister et al. [3]; eugenol, methyl chavicol, carveol and caryophyllene by Datt et al. [2]; eugenol, methyl eugenol and caryophyllene by Lawrance et al. [12]; eugenol, methyl chavicol, linalool and 1:8 cineole by Satti et al. [15]; Gibberellic acid (GA_3) treatment of the crop of *O. sanctum* has been reported increasing the essential oil as well as eugenol content [6]; Harvesting the crop at full bloom

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possessed maximum eugenol content [7] [1] [16] [18] and that the oil possessed larvicidal, synergistic and antibacterial/ antifungal properties [1] [17]. In India and Pakistan the oil is used as a substitute for oil of cloves [8].

Essential oils were extracted, separately, from the plants of *O. sanctum* distributed at Rishikesh and from the plants raised at Haldwani (another place on the foot hills of Himalayas) from the seeds of Rishikesh plants. Chemical composition of the two oil samples were studied in order to know the difference, if any, in their composition due to change of place of distribution/cultivation.

EXPERIMENTAL :

Essential oil was obtained by steam distillation of the fresh plant material. The oil was found mostly concentrated in leaves and flowers. Stalks contained negligible oil. Oil content in leaves and flowers at various phases of plant development are given in table.1.

TABLE 1

OIL CONTENT IN LEAVES & FLOWERS AT DIFFERENT STAGES OF PLANT DEVELOPMENT AT RISHIKESH

Stage of growth	part of the plant	oil content (fresh weight basis)
Pre flowering (immature inflorescence)	leaves	0.65%
	flowers	0.60%
Flowering	leaves	0.58%
	flowers	0.41%
Post flowering	leaves	0.56%
	flowers	0.41%

Physico-chemical properties of the essential oil obtained from the plants (flowering stage) distributed at Rishikesh were:

Sp. gravity/20° = 0.9498 ; Ref. index/20° = 1.5184 ;
Sp. rotation 20° = - 6° 20' ; Acid value = 1.2 ; Ester value = 24.1 ;
Carbonyl value as C_{10} $H_{16}0$ (Oximation) = 1.5 ;
Phenolic content = 66.2 and solubility in 80% alcohol = 1:0.60.

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Chemical composition of the essential oil was studied by the usual derivatization, chromatographic processes like TLC & GLC and IR, UV, PMR and Mass spectroscopy. 50 ml of the oil was shaken thoroughly with 5% aqueous sodium hydroxide followed by 1% sodium hydroxide. The oily layer was separated from the reaction mixture by extraction with petroleum ether (40-60°) and the non-phenolic constituents (16.8 ml) were obtained by distilling off the solvent from the ether extract. The aqueous sodium hydroxide extract was acidified with 20% phosphoric acid to Congo red point and extracted with ether. Ethereal extract was dried over anhydrous sodium sulphate and on distilling off the solvent 32.5 ml phenolic contents were obtained. Constituents present in phenolic and non-phenolic fractions were investigated separately.

PHENOLIC FRACTION :

TLC examination over silica gel G layers under different solvent systems and GLC indicated this fraction to be a single compound. The fraction possessed $d_4^{20} = 1.0616$; $n^{20}_D = 1.541$ and formed phenyl urethane m 97°, benzoate m 69°and tetrabromide m 118° [10] [9] [4]. These properties and IR spectra confirmed this fraction to be eugenol.

NON-PHENOLIC FRACTION :

This fraction was subjected to further separation -

a) Separation of alcohols :

The alcohols from the non-phenolic fraction (16.8 ml) were separated by refluxing the fraction with calculated amounts of 3:5 Dinitro benzoyl chloride and pyridine. The reaction mixture was cooled and water was added. The solid derivative was separated and dried.

TLC examination of the solid derivative over silica gel G coated plates, using benzene : petroleum ether (1:1) as solvent system, butter yellow as reference compound and ethanolic α - naphthyl amine solution (1%) as spray reagent showed the presence of linalool ($R_B = 1.75$), α- terpineol ($R_B = 1.4$) and citronellol ($R_B = 1.65$). Identity of these compounds was further confirmed by Co - TLC.

b) Alcohol free non-phenolic fraction :

The fraction (12.5ml), left after distilling off the pyridine under reduced pressure, was separated quantitatively into 'Hydrocarbon fraction' and

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'Oxygenated fraction' by using a column packed with silicic acid, and n-hexane and ethyl acetate, respectively, as eluting agents [11].

Hydrocarbon fraction :

TLC examination of the fraction over silica gel G coated plates, using n-hexane as solvent system, limonene as reference compound, fluorescein-bromine as reagent for visual identification showed the presence of three compounds viz. β - elemene (R_L = 1.3), β - caryophyllene (R_L = 1.4) and terpinolenc (R_L = 1.5). Identity of these compounds was confirmed by Co-TLC, and PMR/Mass spectroscopy of the compounds obtained from preparative chromatography of the 'Hydrocarbon fraction' [13] [14] [19] [20].

Oxygenated fraction :

TLC examination of the fraction over silica gel G layers using petroleum ether - pyridine (95:5) as solvent system, thymol as reference compound, suitable spray reagents viz. $SbCl_5$:CCl_4 in 1:4 for ethers and fluorescein - bromine for esters for visual identification showed the prescence of methyl eugenol (R_T = 2.3), methyl iso-eugenol (R_T= 2.2) and eugenol acetate (R_T = 2.5). Prescence of these compounds was confirmed by Co-TLC. Identity of the compounds was finally confirmed by PMR/ Mass spectroscopy of the compounds obtained from preparative chromatography of the 'Oxygenated fraction' [13] [14] [19] [20].

GAS CHROMATOGRAPHY OF THE OIL :

Chemical composition of the oil was also studied by Gas chromatography using Perkin Elmer Model 3920. In order to know the difference in chemical composition, the two oil samples viz. from the plants distributed at Rishikesh and from the plants raised at Haldwani, were analysed by Gas chromatography under similar parameters. Different constituents were identified by 'injection method' and comparison of retention time under the same operating parameters using pure compounds. The conditions maintained were :

Column	3.6m long, 2.0 mm I.D.,
Stationary phase	-UCCW 10% in chromosorb, 80-100 mesh.
Column temperature	70-200°C ; @ 2° per minute.
Chart speed	5 mm per minute.
Detector & injector	

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temperature	.....	250° C.
Attenuation	.....	512 & 256.
Range	.....	10
Sample size	.....	4µl-6µl (1:30 solution in $CHCl_3$).

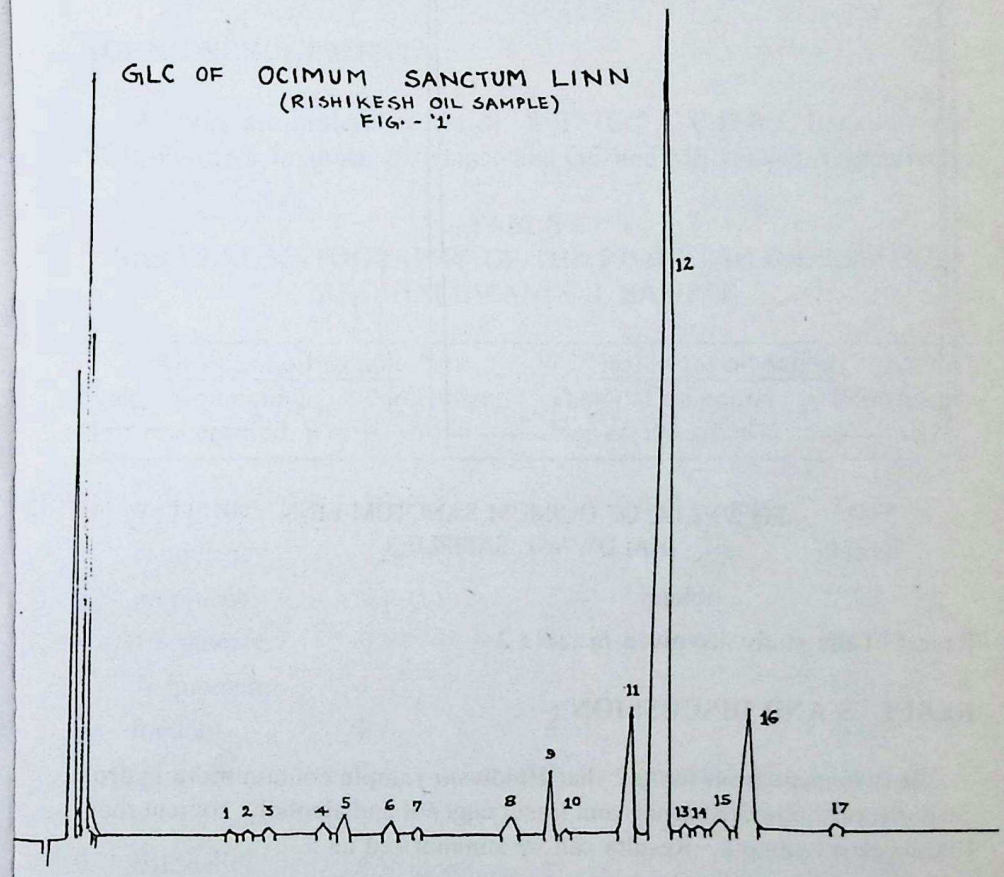

Fig 1 : GLC OF OCIMUM SANCTUM LINN
(RISHIKESH OIL SAMPLE)

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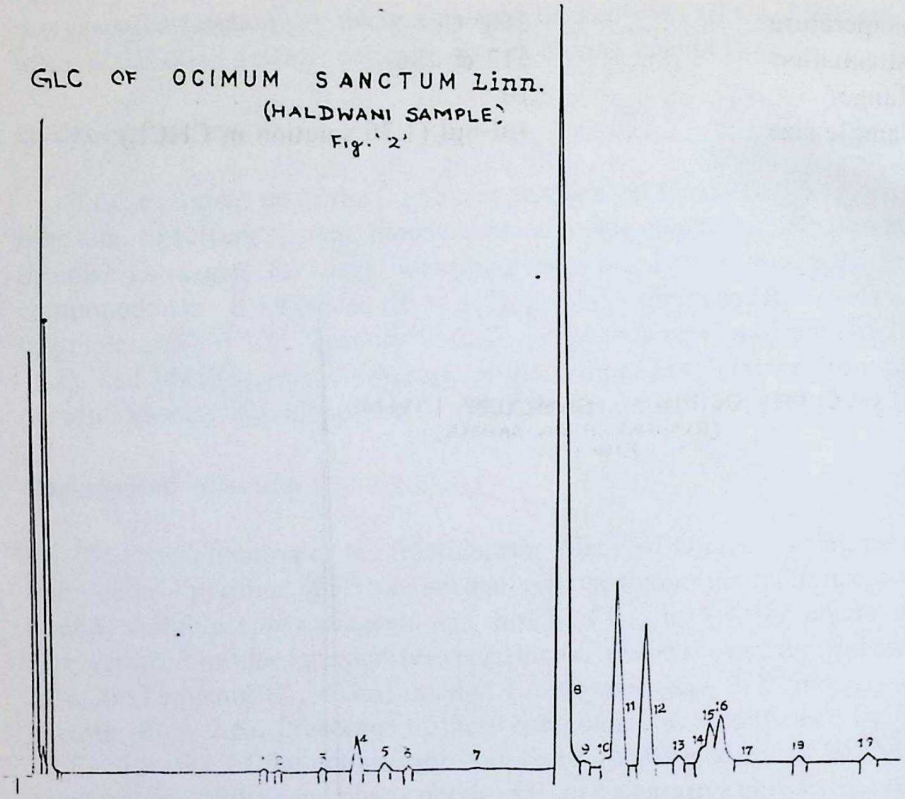

Fig 2 : GLC OF OCIMUM SANCTUM LINN
(HALDWANI SAMPLE)

Result of the study are given in table 2.

RESULTS AND DISCUSSION :

It is evident from table 2 that Haldwani sample contain more hydrocarbons, phenolic derivatives and lesser eugenol and alcoholic content than Rishikesh oil sample. Results can be summarised as :

	Haldwani sample	Rishikesh sample
Hydrocarbons	20.72%	12.85%
Phenolic derivatives	24.00%	0.6%
Eugenol	61.4%	66.5%
Alcoholic content	1.72%	7.3%

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Difference in chemical composition of two oil samples may be attributed to the change of locality/place of cultivation of the plant. Further work is needed to ascertain the actual factors responsible for the difference. The authors have identified twenty compounds in the oil of *O. sanctum* as against nine compounds viz. linalool, geraniol, citronellol, cineole, methyl chavicol, eugenol, carveol, methyl eugenol and caryophyllene, reported as a result of work carried out by number of workers mentioned in literature [2] [3] [5] [12] [15]. Further, Rishikesh oil sample contanining 66.5 percent eugenol can serve commercially as a substitute for Oil of cloves.

ACKNOWLEDGEMENTS :

Authors are grateful to S.C.S. & T. (U.P.); C.D.R.I. Lucknow and NCL, Pune for financial assistance and GC and MS studies, respectively.

TABLE 2
GAS CHROMATOGRAPHY OF THE RISHIKESH OIL SMAPLE AND HALDWANI OIL SAMPLE

Rishikesh oil sample			Haldwani oil sample		
Peak No.	Compound identified	Percentage	Peak No.	Compound identified	Percentage
1.	α- pinene	Trace	1.	1:8 cineole	Trace
2.	camphene	Trace	2.	-	Trace
3.	α- pinene	0.6	3.	linalool	0.12
4.	1:8 cineole	0.6	4.	borneol	1.6
5.	Terpinolene	1.0	5.	-	Trace
6.	linalool	0.8	6.	-	Trace
7.	camphor	0.1	7.	geraniol	Trace
8.	α - terpineol	1.2	8.	eugenol	61.4
9.	citronellol	4.9	9.	-	Trace
10.	geraniol	0.4	10.	-	Trace
11.	bornyl acetate	11.9	11.	methyl eugenol	20.15
12.	eugenol	66.5	12.	β-caryophyllene	17.4
13.	-	Trace	13.	humulene	0.68
14.	-	Trace	14.	methyl isoeugenol	1.65

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15.	methyl eugenol	0.6	15.	aceto eugenol 2.2
16.	β-caryophyllene	11.1	16.	β-elemene 2.1
17.	β - elemene	0.15	17.	γ-elemene Trace
18.	cadinene	0.52		
19.	-	0.56		

-denotes in this table the unidentified peaks.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 11-22

PHARMACOGNOSTIC STUDIES IN LEAF OF *LENOTIS NEPETAEFOLIA* L.

S. Jelani* & M. Prabhakar*

(Received 05-06-1991)

ABSTRACT

The paper deals with Pharmacognosy of *Leonotis nepetaefolia* leaf including its morphological, anatomical, chemical constituents and powder analysis. Contrary to the earlier reports the leaves are amphistomatic, stomata are mostly diacytic, few isotricytic. Four types of trichomes viz. unicellular conical hairs, uniseriate conical hairs, uniseriate capitate hairs, uniseriate peltate hairs have been recorded. The venation is Craspido-brochidodromous which is a new pattern. Midrib consists of one large semicircular shaped and petiole with five, one median large arch shaped and four laterals, two on each of the arms of median bundles. Powder analysis microscopically show fragments of epidermis, mesophyll, trichomes, tracheary elements. Positive tests for glycosides, alkaloids polyphenolases, flavones, juglones are recorded.

Keywords : *Leonotis nepetaefolia.* Pharmacognosy.

INTRODUCTION

The leaves of *Leonotis nepetaefolis* are curative and consider to be emmenagogue februfuge, depurative, narcotic, bitter and laxative: used in skin diseses, amenorrhoea and fevers [1,2,16]. The plant is medicinally used by tribe mundas of Chotanagpur. When a mother's breasts are well and milk does not pass through the nipples, the leaf, juices is also expressed and taken with lime juice and run as a febrifuge [9].

MATERIAL AND METHODS

Twigs of *Leonotis nepetaefolia* were collected from plants growing in Osmania University campus, Hyderabad, Andhra Pradesh. The methodology and terms used are after Prabhakar and Ramayya [18]. Leelavathi et al. [11] Prabhakar and Leelavathi [6]

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Abbreviations used in the text are : Ab: abaxial surface of the organ, Ad: adaxial surface of the organ, D: diameter, Dist: distribution, E.C.F.: epidermal cell frequency, L/W: length/width, S/F.: stomatal frequency.

OBSERVATIONS AND DISCUSSION

Vernacular names : BENGAL: RAJURCHEI; BOMBAY: Matijer, Matisul; BRAZIL; Cordao de frade; CEYLON: Kasitumpai; GUJARATI: Matizer, Matisul, HINDI: Bejurchei, MARATHI: Dipmal, Ekri, MUNDAR: Agaiajanum, Gharia, Matsuenggelsui, Singgelsui, PORTO RICO: Rascamono, SANTALI: Daredhompo, Jonumdhompo; SIMHALESE; Mahayakwanassa; TELUGU; Rana beri; Hanumantabira; Mulagolimedi.

Morphology: Erect. branched, pubescent, aromatic, herb, upto 2 m high, Root: Tap root Stem: branched, obtusely quandrangular with ridged angles, finely pubescent. Leaves : Simple, opposite-deccusate, ovate 10-13 x 7-9 cm, margin serrate, apex acute, pubescent. Petiole: Four to five cm. filifrm, pubescent. Inflorescence: Axillary, verticillaster, globose, about 6 cm in diameter. Flowers: Small, orange-scarlet, sessile, Bract : linear, 6 mm long strongly spinous, pointed, pubescent, Bracteoles: Two bract-like, 1-1.5 mm. Calyx : Somewhat bilabiate; tube 3 m long; lobes subequal, subulate, 2 mm long. Corolla: Petals 5, bilipped; tube 9 mm, glabrous below, densely clothed above with scarlet orange hairs; upper lip 11 mm; lower lip 3 lobed 6 mm, lobes oblong. Stamens: Four didynamous, filaments subequal, 4-6 mm, anther 2-celled, cells divaricate, 1.25 mm. Ovary: Truncate, deeply 4-lobbed, pubescent, ovule solitary in each locule on axile placentation. Style: Filiform 1.5 cm glabrous. Stigma: Subequally bilobed. Fruit : Aggregate of 4 nutlets, nutlets oblong, 3-gonous, Flowering and fruiting : October to December.

Microscopic characters

Surface view of leaf : Anticlinally walls of laminar epidermal cells were described to be wavy to sinuate [5]. Presently the epidermal cells are polygonal anisodiametric and rarely polygonal linear. Anticlinal walls are sinuate sinuses being "V" shaped on abaxial but wavy to sinuate on adaxial sides. In none of the Lamiaceae taxa papillate epidermal cells were reported [5,10,15,19], however, presently papillated epidermal cells are observed on both abaxial and abaxial surfaces which is an important diagnostic charcter and has been reported only in few angiospermous taxa. Anticlinal walls are slightly thick and cytoplasmic contents are scanty. Epidermal cells are

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irregularly arranged and variously oriented Dist : All over the leaf lamina except on veins (Figs. 1&2).

Foliar costal cells and stomatal distribution patterns were considered to be of identification value [1]. The costal cells in the present taxon are mostly polygonal linear and a few rectangular linear. Anticlinal walls are straight and slightly thick. The costal cells are parellely oriented and irregularly arrange. Dist : On all grades of veins.

Type of the stomata in *Leonotis nepetaefolia* was reported to be diacytic [5,10,15,19,20]. However, presently the dominant diacytic stomata are found to be associated with few isotricytic stomata on abaxial side. Stomata are monocyclic. (Fig. 1&2). Shape of the guard cells are elliptic. Size of the stomata on adaxial and abxial surface range from 23-18-15 µm in (length) and 15-12-11 µm (width). Subsidiaries are mostly "a" type rarely of "f" type and are indistinct from epidermal cells (Fig. 1&2). Dist :The leaves were reported to be hypostomatic by Inamdar and Bhatt [5] however presently they are observed to be amphistomatic and distributed all over the leaf lamina except on veins of both the surfaces (Figs. 1&2).

In the past glandular (uni or bicelled head) and non glandularl (uniseriate and multicellular) trichomes were reported [5,10,15,19,20]. However there were no reports on presence of unicellular conical hairs which are presently observed. Besides other types like uniseriate conical hairs, uniseriate capitate hairs, uniseriate peltate hairs are also observed which are described below:

1. Unicellular conical hair : Foot: consisting of the basal end of the hair, indistinct, contents slightly dense; walls thin. Body : short or long, conical, apex acute, contents scanty; walls thin; surface smooth (Fig. 6).

2. Uniseriate conical hairs : Foot : One - celled, embedded or slightly projected above the epidermis, indistinct, contents scanty, walls thin. Body: Conical unicellular to uniseriate, 1 to 3 - celled in length; cells longer than broad contents scanty, walls thin, surface smooth (Fig. 7).

3. Uniseriate capitate hairs : Foot : One-celled, embedded, or slightly projected above the epidermis indistinct, usually broader than long; contents scanty; walls sthin Stalk : one to three celled; two tired; broader than long or squarish, contents scanty; walls thin surface smooth. Head : Eight celled, cells longer than broad; contents dense; walls thin surface smooth (Fig. 8).

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4. Uniseriate peltate hairs : Foot : One-celled embedded or slightly projected above the epidermis, indistinct, usually broader than long; contents scanty; walls thin. Stalk : One to three-celled, two tired, broader than long or squarish, contents scanty; walls thin surface smooth, Head : Four or eight celled, cells longer than broad; contents dense; walls thin; surface smooth (Fig. 9).

Foliar venation as taxonomic tool has been since long time in use [16] which is equally important in Pharmacognostic studies [6-8,12]. Presently it is observed that most of the secondaries in *Leonotis* show branching. Further the basal secondaries terminate at margins (like in Craspidodromous pattern) and distal secondaries join the superadjacent secondaries forming loops (as in Brochidodromous pattern). This pattern of venation do not fit under any of the pattern described by Hickey [4] or Melville [14]. Hence presently described as a new pattern Craspido-brochidodromous which can be used as significant character for identification of the foliar drugs. (Fig. 3). Secondaries are six pairs, sub-opposite to opposite to alternate. The proximal loop forming branches joing the super adjacent secondaries at right angles to obtuse angle. Tertiaries are percurrent and their angle of origin is acute: right angle (AR). The relationship of the tertiaries to midvein is oblique to perpendicular and their course is retroflexed and predominantly opposite. Higher order is observed upto 5° vein.Areoles are polygonal, regular in orientation and 600/cm^2. Veinlets are simple and branched once, veinilets are 0-1/areole and 500/cm^2 (Fig. 5).

Sectional view of leaf : In transaction of leaf lamina is ribbed abaxially at primary, secondary and tertiary veins but adaxially inconspicuously ribbed only at primary veins (Fig. 3). Epidermal cells are circular to oval shaped, abaxial ones are smaller in size. (on lamina wings Ad: L/W=41-25-19/19-14-11; Ab:L/W = 30-21-15/15-13-11; midrib D: 19-15-11/16-28-8 μm) with thin cuticle and scanty contents. Guard cells with upper ledges. Palisade one layered. Cells are cylindricl and loosely arranged, (L/W: 79-57-49/23-15-11 μm). Near midvein and major lateral veins on both sides, hypodermis consists of 1-4 layered compactly arranged angular collenchyma (D=27-17-11 μm) and rest the ground tissue is parenchymatous. Parenchyma is 3 to 4 layers; cells polygonal to circular in shape(D-46-33-27 μm). Midvein consists of central large, semi-circular shaped single vascular bundle which is collateral and apericyclic. (Figs. 3). Tracheary elements are polygonal to circular (D=19-13-8 μm) more in midvein and secondary veins; few in others, arranged in rows. Secondary wall thickenings annular (free) helical (helices single or double). Perforation plate is simple.

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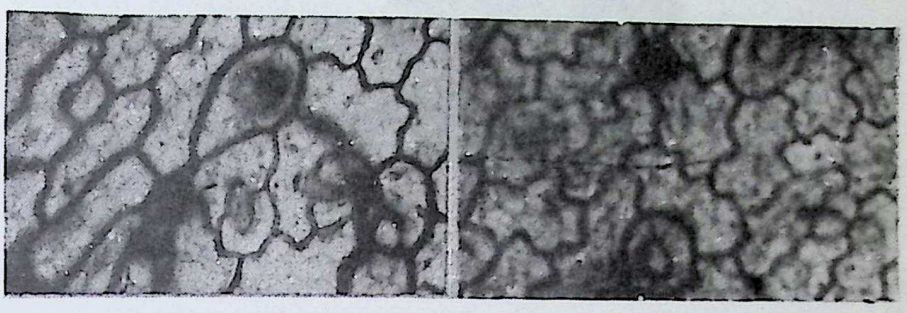

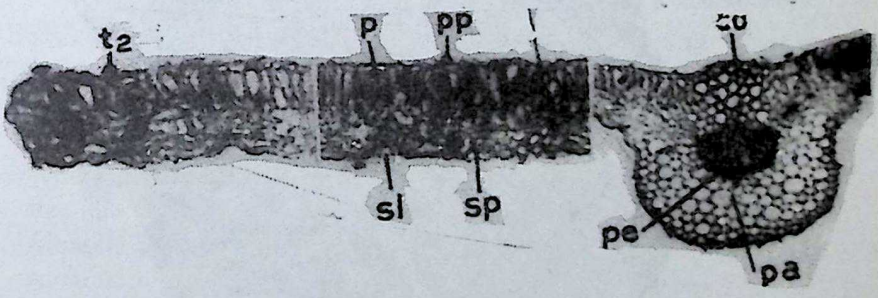

Fig. 1-3

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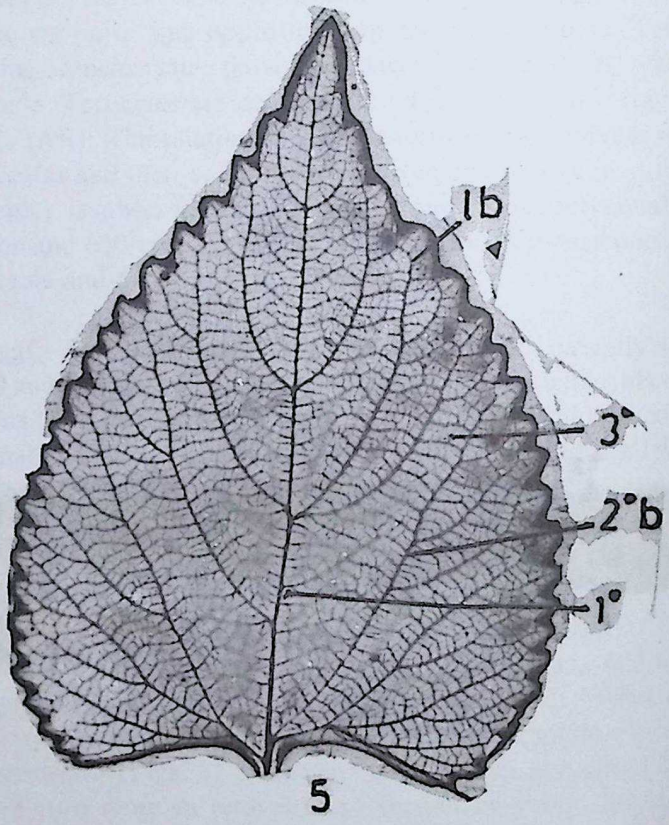

Fig. 4 & 5

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In transection the petiole is circular and adaxially groved (Fig. 4). Epidermal cells are circular to oval in shape (D=15-13-11 μm). Ground tissue is heterogenous with 2 to 3 layered compaetly arranged collenchymatous hypodermal circular to polygonal.cells (19-17-15 μm) with scanty contents. Rest of the ground tissue is 5 to 8-layered parenchyma. Cells are polygonal (D=87-66-46 μm) with thin walls and scantly contents. Vascular bundles are 5; the median one is large, lunar shaped and the four small laterals, two on each of the arms of median bundle (Fig. 4). Tracheary elements are polygonal to circular (23-17-13 μm) arranged in rows. Secondary wall thickenings are annular (Free), helical (helices single or double) with simple perforation.

Powder Study

The powder is light brown, aromatic. Microscopic examination of the powder revealed the following elements 1) Fragments of epidermis with wavy or sinuate anticlinal walls and diacytic stomata (Fig. 10); 2) Fragments of costae or petiole showing small polygonal epidermal cells without stomata (Fig. 11); 3) Full or bits of uniseriate conical capitate, and peltate hairs (Fig. 12); 4) groups or isolated palisade, spongy and ground parenchyma and collenchyma (Fig. 13-16); 6) bits of vessels with annular, hellical thickeings (Fig. 17).

Besides the microscopic observations the cold extracts of the powder in different solvents and the moist and dry powders are observed under ordinary light and Ultraviolet light to record the colour changes. (Colours were compared with colour index [13] The details are given in table.

OBSERVATION OF LEAF UNDER ORDINARY AND U. V. LIGHT

Sl. No. Powder	Ordinary Light	U. V. light
1. Dried powder	CB	LB
2. Water extract	RM	DG
3. Residue	DP	GB
4. Alcholic extract	GY	GB
5. Alcholic residue	GY	DP
6. Chloroform extract	OB	GB
7. Chloroform residue	BB	OB

BB. Brown black; CB. clay brown; DG. dark green; DP. dull brown; GB. greenish brown; GY. greenish yellow; LB. light brown; OB. olive brown; RM. ripe mango colour.

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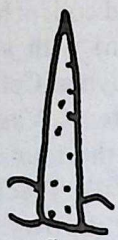

Fig. 6

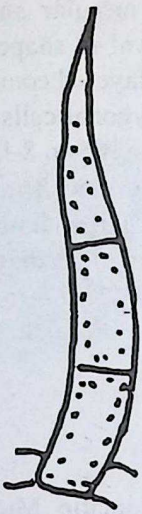

Fig. 7

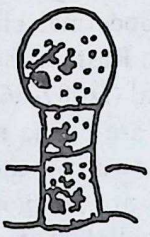

Fig. 8

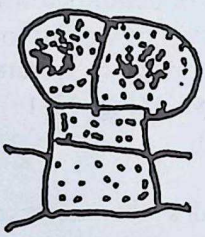

Fig. 9

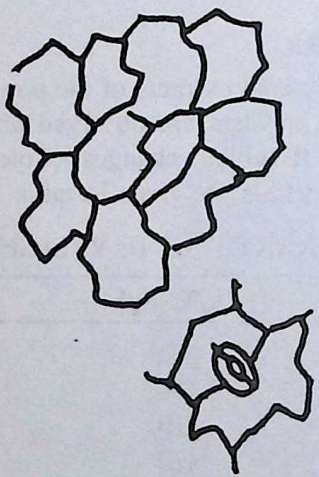

Fig. 10

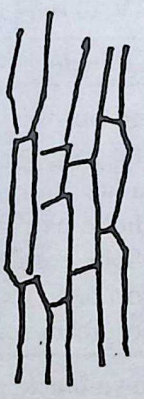

Fig. 11

Fig. 6-11

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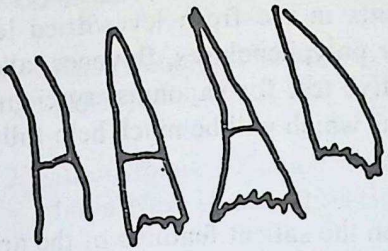

Fig. 12

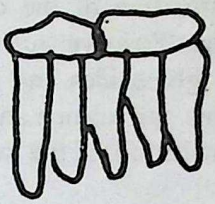

Fig. 13

Fig. 14

Fig. 15

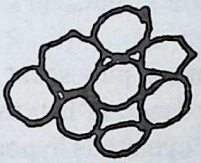

Fig. 16

Fig. 17

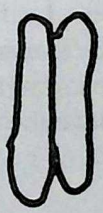

Fig. 18

Fig. 12-17

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Following the simle phytochemical tests recommended by Gibbs [3] for identification of the chemical constitutents in the fresh leves/dried leaf powder. Positive tests were recorded for polyphenolases, flavones, alkaloids, glycosides and juglones, and negetive test for saponins, syrengins, acubins and aurone and leucoanthocyanins which will be much help full in identification of the taxa in powder form.

In the light of the present investigation the salient features of the drug which would be useful in identification in its different states are follows :

Leaf : Leaves ovate, base sub-cordate, apex acute; margin serrate, densely hairy, green 10-13 x 7-9 cm with Craspido-brochidodromous venation.

Secondary veins are six pairs, mostly branched. Epidermal cells are mostly polygonal anisodiametric, sides abaxially sinuate, sinuses being "v" shaped and adaxially wavy to sinuate. Costal cells distinct on all veins. Leaves are amphistomatic; stomata mostly diacytic rarely associated with anisotricytic. Unicellular and uniseriate conical hairs uniseriate capitate hairs, uniseriate peltate hairs present all over except on margin. In transection leaf is bifacial, abaxially ribbed at midvein, secondary vein, tertiary vein and feebly at adaxial midvein. Palisade is one layered and spongy mesophyll is 7-8 layered. Midvein with single large semi circular shaped vascular bundle.

Petiole : Five cm long, slender, pubescent, In transection it is circular and adaxially groved. Ground tissue is heterogenous consisting of 3-layered, compactly arranged collenchymatous hypodermis and 5-8 layered parenchymatous tissue. Vascular bundles are five; the median one is large, lunar shaped and the four small laterals two on each of the arms of median bundle.

Leaf powder : Leaf powder is brown, aromatic microscopically fragments of epidermis, stomata, mesophyll, trichomes and tracheary elements can be observed. Following the simle tests [3] positive test for glycosides, polyphenolases, alkaloids flavounes, juglones are recorded.

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EXPLANATON TO FIGURES

Plate 1. *Leonotis nepetaefolia* L. Fig. 1 & 2. Adaxial and abaxial epidermis of leaf lamina (x 440); 3. Transection of leaf lamina from midrib (x 52), lamina (x96) and margin (x 102); 4. Sectional view of petiole (x 75); 5. Venation pattern of leaf lamina. (CO. Collenchyma; 1b. loop forming branch; Pe. Phloem parenchyma; Ph. Phloem. pp. Papilate: Sc. Secretory cell ; Sp. Spongy parenchyma X. xylem; 1°. Primary vein; 2°b. Secondary vein branched. 3°. Tertiary vein).

Figs. 6. Unicellular conical hair 7. Uniseriate conical hair; 8. Uniseriate capitate hair. 9. Uniseriate peltate hair; 10. Pieces of the epidermal cells; 11. costal cells; 12. Full or bits of trichomes; 13 & 18. Palisade parenchyma; 14. Spongy parenchyma; 15. Collenchyma cells; 16. Ground parenchyma; 17. Tracheary elements.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 23-30

THERMODYNAMICS AND STOCHASTICS OF BIOLOGICAL GROWTH

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(Received 19-01-1991 and after revision 28-06-91)

ABSTRACT

A non-equilibrium thermodynamic model of biolgical growth together with the stochastic analysis of nonequilibrium fluctuation has been presented.

AMS (MOS) Subject classification (1980) : 92A7

Keywords : Gompertz equation, Entropy production, Quasi-stationary state, Non-equilibrium fluctuation.

INTRODUCTION

The study of growth, development and aging of living organism is an important field of biological sciences. A purely thermodynamic theory of growth and aging is inadequate in view of the stochastic nature due to internal noises irrespective of the random effect of the environment. The non-equilibrium thermodynamic model combined with the stochastic theory of non-equilibrium fluctuation is the proper physico-chemical theory of biological growth and aging [1]. In some earlier papers [2-4] we have tried to develop non-equilibrium thermodynamic and stochastic models of biological growth and aging. In the present paper we shall study the thermodynamic and stochastic behaviour of a growth model which was earlier derived from stochastic thermodynamic model of the system.

BIOLOGICAL GROWTH : THERMODYNAMIC AND STOCHASTIC MODELS

The living system is an open system. For the applicability of the thermodynamics of irreversible processes to the process of biological growth and development, we must have the entropy balance equation for open system.

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$$\frac{1}{m}\frac{dS}{dt} = \frac{1}{m}\frac{d_i S}{dt} + \frac{1}{m}\frac{d_e S}{dt} \tag{1}$$

where S is the entropy of the system at any time t, $d_i S$ is the entropy production due to irreversible processes inside the system and $d_e S$ is the flow of entropy due to the exchange with the environment. A fundamental quantity of non-equilibrium processes is the dissipative function ψ which is related to the rate of entropy production by the relation

$$\psi = T\frac{d_i s}{dt} \tag{2}$$

where T is the temperature of the system. The basic problem of the non-equilibrium thermodynamics of biological growth and development is to find the appropriate dissipative function for the living system. In the earlier paper [2] we have derived the expression of the dissipative function ψ as

$$\psi = A\, e^{-\alpha t} \tag{3}$$

where A is a constant and α is a parameter. Following Zotin and Zotina [1], let us take the specific current weight as the thermodynamic flux

$$\frac{1}{m}\frac{dm}{dt} = J \tag{4}$$

where m is the growth parameter (weight or mass) of the system. If X be the thermodynamic force corresponding to the flux J, then we have [5]

$$\psi = JX = A\, e^{-\alpha t} \tag{5}$$

Using the linear phenomenological relation $J = LX$, we get finally

$$\frac{dm}{dt} = a\, e^{-\beta t} m \tag{6}$$

where $a = (LA)^{1/2}$, and $\beta = \alpha/2$ is the senescence parameter.

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The equation (6) is the famous Gompertzian equation which describes a wide variety of growth such as tumour growth, embryonic growth, growth of animals and men [6].

Let us now study the stochastic behaviour of the Gompertz growth process described by the equation (6). For this let us extend the deterministic equation (6) to the form of the stochastic differential equation.

$$\frac{dm}{dt} = a\, e^{-\beta t} m + f(t)$$

$$= A(t)m + f(t) \qquad (7)$$

where $A(t) = a\, e^{-\beta t}$ is the time - dependent drift coefficient and $f(t)$ is the random perturbation which is assumed to be a whitenoise of intensity ε satisfying the conditions

$$< f(t) > = 0 \text{ and } < f(t_1)\, f(t_2) > = 2\, \varepsilon\, \delta\, (t_2 - t_1) \qquad (8)$$

where $< >$ represents the average over the stochastic process. The equation (7) is generalization of the famous Langevin equation with time - dependent drift coefficient.

The stochastic behaviour of the process described by equation like (7) is manifested in the moment equations or moments, particularly the second - order moment of the growth parameter $m(t)$. The non-stationary stochastic behaviour of the stochastic Gompertzian equation (7) has been studied elsewhere [4]. In this paper we are, however, interested in the quasi-stationary state of the system. Such a state is characterized by the slow variation of the drift coefficient $A(t)$ and the diffusion coefficient ε which is assumed here to be a constant. The stationary (or steady) non-equilibrium state of living systems occur at different times for differtent living organisms. As far example, for human it generally appear about the age 25 to 30. During the steady non-equilibrium state or homeostasis the growth parameter remains practically constant, but at the quasi-stationary and non-stationary states it is subject to fluctuation due to innumerous internal random effects irrespective of the external or environmental influences. The study of the stochastic behaviour or fluctuations of the growth parameter for quasi-stationary state which is very near to the stationary non-equilibrium is of great biological importance [7].

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The mean square fluctuation of the growth parameter $m(t)$ satisfying the stochastic differential equation (7) is given by a generalized form of the famous Nyquist's relation [2].

$$<m^2(t)>=\frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\left|\frac{A(t)}{(A(t)+i\omega)}+\frac{A(t)}{(A(t)+i\omega)^3}\right|^2 d\omega \qquad (9)$$

Using $A(i) = a\, e^{-\beta t}$, $A(t) = -\beta A(t)$ and evaluating the integrals appearing in (9), we have

$$<m^2(t)>=\varepsilon\left[\frac{1}{A(t)}-\frac{\beta}{2A^2(t)}+\frac{3\beta^2}{8A^3(t)}\right] \qquad (10)$$

Let us now use the condition of quasi-stationarity. For the problem under consideration this is satisfied as stated earlier by the criteria of slow - variation of the drift - coefficient $A(t) = a\, e^{-\beta t}$ i.e. by the the condition

$$\left|\frac{A(t)}{A(t)}\right| << 1 \text{ or } |\beta| << 1 \qquad (11)$$

Neglecting terms containing β and its higher powers, we have

$$<m^2(t)>=\frac{\varepsilon}{A(t)}=\left(\frac{\varepsilon}{a}\right) exp\ (\beta t) \qquad (12)$$

showing that the fluctuation of the growth parameter (mass or weight) increases exponentially with time unless there is some restriction on the order of (βt). Apart from the fact the fluctuation becomnes infinitely large, there is another difficulty with the Gompertzian model equation (6). The equation (6) shows that the stationary or steady growth of the living system is reached after an infinitely large time. This is practically impossible for living organisms having terminated growth and finite life span. All these show that there must be some restriction on the order of the exponent (βt). In the next section we are going to resolve all these difficulties faced by Gompertzian Model.

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A LINEAR GROWTH MODEL: THERMODYNAMICS AND STOCHASTICS

To resolve the difficulties faced by the Gompertz equation (6), we assume that

$$|\beta t| << \tag{13}$$

Under this condition, the equation (6) is approximated to

$$\frac{1}{m}\frac{dm}{dt} = J = a\beta\left[\frac{1}{\beta} - t\right] = L_0(T_m - t) \tag{14}$$

where $1/\beta = T_m$ is the maximum age, $L_0 = a\beta$ is the phenomenological or kinetic coefficient. If we take $= T_m - t$ as proportional to the thermodynamic force corresponding to the flux J, the relation (14) becomes the linear phenomenological relation between the thermodynamic flux and force. The linear approximation of (6) neglecting quadratic and higher powers of (β t) is justified by two facts. The first is that the linear approximation makes the equation amicable to the theory of linear thermodynamics of irreversible processes and the secondly it removes the infinite time to reach the steady state of the growth and also the infinitely increasing of fluctuation as we shall see a bit latter. Above all the linear model (4) is able in explaining a wide variety of biological phenomena related to growth and development of living organism [1]. It is infact, the starting or basic equation in an early development of non-equilibrium thermodynamics of biological growth by Zotin and Zotina [1]. In this paper we have arrived at it from a more general setting of thermodynamic model of growth of living organism. In terms of the growth parameter m, the equation (14) can be written

$$\frac{dm}{dt} = a\beta(T_m - t)m \tag{15}$$

Integrating (15) and using the initial conditions

$$t = 0,\ T_m = 0,\ m = m_0 \text{ (say)} \tag{16}$$

we have

$$(T_m - t)^2 = \frac{2}{a\beta}\log\,(m_0 / m) \tag{17}$$

expressing the thermodynamic force in terms of the growth parameter (mass or weight) of the system.

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The rate of entropy production then become

$$\frac{d_i S}{dt} = \frac{1}{T} JX = \frac{a\beta}{T}(T_m - t)^2 \qquad (18)$$

Integrating, we have

$$\Delta_i S = (S_i)_f - (S_i)_0 = \frac{a\beta}{T}\int_0^t (T_m - t)^2 dt \qquad (19)$$

Using (17) and (18), the entropy production from the initial state O at time O to the final state f at time t can be expressed as

$$\Delta_i S = \frac{2}{T}\int_{m_0}^{m} (\log m_{\,o} - \log m) dm$$

$$= \frac{2}{T}\left[(m - m_0) - m \log (m / m_{0})\right] \qquad (20)$$

which expresses the entropy production in terms of mass or weight of the system, the temperature T being assumed to be constant during the time $(0,t)$. The expression (20) of entropy production has important sigificance. Its mathematical derivation including its biological implication will be studied later on.

Let us now consider the stochastic behaviour of the linear equation (15). The stochastic extension of (15) is given as before by

$$\frac{dm}{dt} = a\beta(T_m - t)m + f(t) \qquad (21)$$

Using either directly the formula (9) with $A(t) = a\beta(t_m - t)$ or the relation (12) with the restriction (13) (i.e.$|\ \beta t\ | << 1$) the second - order moment of $m(t)$ is given by

$$< m^2(t) >= \frac{\varepsilon}{a} \qquad (22)$$

Thus, in contrast to the case of Gompertz model, the fluctuation does

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not show infinitely diverging character. The above fluctuation, of course, depends on the intensity ε of the noise and the growth rate α.

CONCLUSION

The paper dealing with the thermodynamic and stochastic models of biological growth can be divided into two parts discussed in section (2) and (3) respectively. The first which deals with the thermodynamic derivation of Gompertz growth and its stochastic behaviour is faced with some difficulties as to the infinite time of reaching steady - state and diverging fluctuation. These difficulties have been resolved in the second part (section (3)) which deals with the linear model derived as a linear approximation of Gompertz equation. This linear model was earlier proposed by Zotin and Zotina [1], here we have arrived at it from a general non-equilibrium setting and studied its stochastic behaviour and stressed its importance in relation to the Gompertzian model of growth of living organism. The whole analysis based on simple mathematical equations or models, there is nothing new numerical or graphical works to be added.

ACKNOWLEDGEMENT

The work was done under a project sanctioned by C.S.I.R. (India). The second author wishes to thank C.S.I.R. for a Junior Research Fellowship. The authors wish to thank the referee whose comments helped the remodelling of the paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp- 31-46

ON NONLINEAR STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS INVOLVING ITÔ-CLIFFORD INTEGRALS

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(Received 04.08.1991)

ABSTRACT

The aim of this paper is to study a general class of nonlinear stochastic integrodifferential equations involving Itô-Clifford integrals. We study the problems of existence, uniqueness and stability of the solutions by using the method of successive approximation.

AMS Subject Classification (1991): 60 H 20

Keywords and Phrases: Stochastic Integrodifferential Equations, Itô-Clifford Integrals, Existence and Uniqueness, Stability, successive approximation.

INTRODUCTION

In a recent paper [1] Barnett, Streater and Wilde have introduced the concept of Itô-Clifford integral and established some basic properties of this integral. They have presented in [2] an interesting study relating to existence, uniqueness and stability properties of the solutions of a stochastic differential equation involving Itô-Clifford integral. This newly formulated stochastic integral and corresponding stochastic equations are yet in their developing stage and awaits for further attention by workers in this field. The aim of this paper is to study the properties of the solutions of a general stochastic integrodifferential equation

$$(1.1)\quad dx(t) = F(t,x(t), A_1x(t), A_2x(t))dt + G(t,x(t), B_1x(t), B_2x(t))\,dw(t) + dw(t)\,H(t,x(t), P_1x(t), P_2x(t))$$

where

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$$A_1 x(t) = \int_{t_0}^{t} f_1(t, s, x(s))\, ds,\ A_2 x(t) = \int_{t_0}^{t} f_2(t, s, x(s))\, dw(s),$$

$$B_1 x(t) = \int_{t_0}^{t} g_1(t, s, x(s))\, ds,\ B_2 x(t) = \int_{t_0}^{t} g_2(t, s, x(s))\, dw(s),$$

$$P_1 x(t) = \int_{t_0}^{t} h_1(t, s, x(s))\, ds,\ P_2 x(t) = \int_{t_0}^{t} h_2(t, s, x(s))\, dw(s),$$

and $w(t)$ is Fermion martingale, with initial condition $x(t_0) = x_0$, $t \geq t_0$ and $x_0 \varepsilon L^2(\ell t_o)$.

As in [2] we consider the case of $L^2(R_+)$, $R_+ = (0, \infty)$ with complex conjugation. Let ℓ denote the w^*-algebra generated by the bounded operators $\{\Psi(z) = C(z) + A(z), z \varepsilon L^2(R_+)\}$ where $C(.), A(.)$ are the Fock fermion creation and annihilation operators respectively and $A(z)$ is the adjoint of $C(z)$; $A(z) = C(z)^*$. For given $t \geq 0$ denote by ℓt the W^* - subalgebra of ℓ generated by the fields $\psi(z)$ for $z \varepsilon L^2(R_+)$ with ess supp $z \subseteq [0, t]$ and the completion of ℓ_t with respect to $\| . \|_p$ by $L^p(\ell_t)$. Therefore, for $1 \leq p < \infty$, $L^p(\ell_t)$ may be considered as a subspace of $L^p(\ell)$. A map $x: R_+ \to L^2(\ell)$ is said to be adapted if $x(t) \varepsilon L^2(\ell_t)$ for each $t \varepsilon R_+$ and a map $Z: (R_+)^2 x L^2(\ell_t) \to L^2(\ell_t)$ is said to be adapted if for any $s \leq t$, $s, t \in R_+$ and $u \in L^2(\ell_t)$ we have $Z(t, s, u) \in L^2(\ell_t)$. Similarly a map $Y: R_+ x [L^2(\ell_t)]^3 \to L^2(\ell_t)$ is said to be adapted if for any $t \in R_+$, $u_i \varepsilon L^2(\ell_t)$, $i = 1, 2, 3$, we have $Y(t, u_1, u_2, u_3) \in L^2(\ell_t)$. In particular if $x_i: R_+ \to L^2(\ell)$, $i = 1, 2, 3$, $z: (R_+)^2 x L^2(\ell) \to L^2(\ell)$ and $Y: R_+ x [L^2(\ell)]^3 \to L^2(\ell)$ are adapted then so are the maps $t \to Z(t, s, x_i(s))$ and $t \to Y(t, x_1(t), x_2(t), x_3(t))$. Let $V_1, \ldots, V_n \varepsilon L^2_{10c}(R_+)$ so that for any $t \geq 0$, $v_i \chi_{[0,t]} \varepsilon L^2(R_+)$, $1 \leq i \leq n$. The Wick-ordered monomial : $\psi(v_1 \chi_{[0,t]}) \ldots \psi(v_n \chi_{[0,t]})$: is denoted by $w(t)$. It is proved in [1., Theorem 2.2, p. 176] that $w(t)$ is an L^∞-martingale adapted to the family $\{\ell_t : t \varepsilon R_+\}$. For detailed notations and definitions see [1,2].

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The general formulation of equation (1.1), which in turn contains as a special case the stochastic differential equation studied by Barnett, Streater and Wilde [2], is motivated by the study of general forms of Itô -type stochastic equations by Berger and Mizel [3], Kannan and Bharucha Reid [4] and Pachpatte [5]. In section 2 we formulate our results on the existence, uniqueness and stability of the solutions of equation (1.1) and its more general form. Section 3 deals with the proofs of our main results. The method of successive approximation is used to establish our results.

STATEMENT OF RESULTS

Before stating the main results in this section we list the necessary preliminaries from [1,2] and the hypotheses needed in our subsequent discussion.

Fix $v_1, \dots, v_n \in L^{\infty}_{10c}(R_+)$ and suppose that $W(t) =: \psi(v_1 \chi_{[0,t)}) \dots \psi(v_n \chi_{[0,t)}):$ for $t \in R_+$. Then $W(s)\, W^*(s) = a(s)1$, where

$$(2.1) \qquad a(s) = \int_0^s \dots \int_0^s |v(s_1, \dots, s_n)|^2 \, ds_1 \dots ds_n,$$

with $v(s_1, \dots, s_n) = (n!)^{1/2} \mathcal{A}(v_n \otimes \dots \otimes v_1)(s_1, \dots, s_n)$ see [1, p. 194]. Let μ denote the measure on R_+ given by the positive increasing function $s \to a(s)$ that is $\mu([\alpha, \beta]) = a(\beta) - a(\alpha)$ for $\alpha \le \beta$ in R_+ and let $f : R_+ \to L^2(\ell)$ be measurable and adapted with $\int_0^t \| f(s) \|_2^2 \, d\mu(s) < \infty$. It is shown in [1] that the Itô-Clifford stochastic integral satisfies the following property,

$$(2.2) \qquad \left\| \int_0^t f(s)\, dW(s) \right\|_2^2 = \int_0^t \| f(s) \|_2^2 \, d\mu(s).$$

Furthermore $\int_0^t dW(s) f(s) = \left(\int_0^t f^*(s)\, dW^*(s) \right)^*$ and $W^*(s)\, W(s) = a(s)\, 1$, therefore we also have

$$(2.3) \qquad \left\| \int_0^t dW(s)\, f(s) \right\|_2^2 = \int_0^t \| f(s) \|_2^2 \, d\mu(s),$$

see [1,2].

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The equation (1.1) is equivalent to the stochastic integral equation

$$(2.4)\quad x(t) = x_0 + \int_0^t F(s, x(s), A_1 x(s), A_2 x(s))\, ds$$

$$+ \int_0^t G(s, x(s), B_1 x(s), B_2 x(s))\, dW(s) + \int_0^t dW(s)\, H(s, x(s), P_1 x(s), P_2 x(s))$$

The integrals $A_1x(s)$, $B_1x(s)$, $P_1x(s)$ are $L^2(\ell)$ Bochner integrals and the integrals $A_2x(s)$, $B_2x(s)$, $P_2x(s)$ are Itô-Clifford integrals, see [2]. Further, the first integral on right in (2.4) is an $L^2(\ell)$ Bochner integral while the second and third integrals are Itô-Clifford integrals. In order these integrals to be well defined, it is sufficient for the maps $s \to f_1(s, \tau, x(\tau))$, $s \to g_1(s, \tau, x(\tau))$, $s \to h_1(s, \tau, x(\tau))$ and $s \to F(s, x(s), A_1x(s), A_2x(s))$, $\tau \le s$ to be L^2 - continuous and the maps $s \to f_2(s, \tau, x(\tau))$, $s \to g_2(s, \tau, x(\tau))$, $s \to h_2(s, \tau, x(\tau))$, $s \to G(s, x(s), B_1x(s), B_2x(s))$ and $s \to H(s, x(s), P_1x(s), P_2x(s))$, $\tau \le s$ are to be adapted and L^2 - continuous. This is the situation when $s \to x(s)$ and f_2, g_2, h_2, G and H are continuous and adapted and f_1, g_1, h_1 and F are contiuous. In order to ensure that the map $s \to x(s)$ is adapted, we assume that f_1, g_1, h_1 and F are adapted. We make use of the above assumption throughout the paper without further mention.

For convenience we list the following assumption :

(H_1) For $x, y, x_k, y_k, z_k \in L^2(\ell)$, $k = 1, 2$ and $t \in [0, T]$ there exist constants K_i, M_i, N_i, $i=1,2$ and K, M, N such that

$$\| f_i(t, s, x) - f_i(t, s, y) \|_2 \le K_i \| x-y \|_2,$$

$$\| g_i(t, s, x) - g_i(t, s, y) \|_2 \le M_i \| x-y \|_2,$$

$$\| h_i(t, s, x) - h_i(t, s, y) \|_2 \le N_i \| x-y \|_2,$$

for all $0 \le s \le t \le T$ and

$$\|F(t, x_1, y_1, z_1) - F(t, x_2, y_2, z_2)\|_2 \le K(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$

$$\|G(t, x_1, y_1, z_1) - G(t, x_2, y_2, z_2)\|_2 \le M(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$

$$\|H(t, x_1, y_1, z_1) - H(t, x_2, y_2, z_2)\|_2 \le N(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$

for all $0 \le t \le T$.

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We now state our main result on the existence and uniqueness of the solution of equation (1.1)

Theorem 1.: Let the hypothesis (H_1) be fulfilled. Then for any $x_0 \varepsilon L^2(\ell t_0)$ there exists exactly one L^2-continuous adapted solution of equation (1.1) on $I = [t_0, \infty)$ with $x(t_0) = x_0$

The result established in Theorem 1 can be extended to the following general form of stochastic integrodifferential equation involving Itô-Clifford integrals driven by $W^{(j)}(t)$, $1 \le j \le r$, of the form

$$(2.5)\quad dx(t) = \sum_{j=1}^{r} F^{(j)}(t, x(t), A_1^{(j)} x(t), A_2^{(j)} x(t))\, dt + \sum_{j=1}^{r} G^{(j)}(t, x(t), B_1^{(j)} x(t), B_2^{(j)} x(t))\, dw^{(j)}(t) + \sum_{j=1}^{r} dw^{(j)}(t) H(t, x(t), P_1^{(j)} x(t), P_2^{(j)} x(t))$$

for $t \in T$ with initial condition $x(t_0) = x_0 \varepsilon L^2(\ell_{t0})$
where

$$A_1^{(j)} x(t) = \int_{t_0}^{t} f_1^{(j)}(t, s, x(s))\, ds,\ A_2^{(j)} x(t) = \int_{t_0}^{t} f_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

$$B_1^{(j)} x(t) = \int_{t_0}^{t} g_1^{(j)}(t, s, x(s))\, ds,\ B_2^{(j)} x(t) = \int_{t_0}^{t} g_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

$$P_1^{(j)} x(t) = \int_{t_0}^{t} h_1^{(j)}(t, s, x(s))\, ds,\ P_2^{(j)} x(t) = \int_{t_0}^{t} h_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

and $W^{(j)}(t)$ denotes the Wick-ordered monomial

$: \psi(v_1^{(j)} \chi^{(j)}_{[0,t]}) \ldots \psi(v_n^{(j)} \chi^{(j)}_{[0,t]}) :$ for $t \in R_+$. We note that $(W^{(j)}(s))\,(W^{(j)}(s))^* = a^{(j)}(s)1$ where $a^{(j)}(s)$ is defined as in (2.1) with corresponding changes. Let $\mu^{(j)}$, $1 \le j \le r$ denote the measure on R_+ given by the positive increasing function

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$s \to a^{(j)}(s)$. We assume that the functions $f_i^{(j)}, g_i^{(j)}, h_i^{(j)}, i = 1,2, j=1,2, \ldots, r$ defined on $(R_+)^2 x L^2(\ell)$ into $L^2(\ell)$ and functions $F^{(j)}, G^{(j)}, H^{(j)}$ defined on $R_+ x[L^2(\ell)]^3$ into $L^2(\ell)$ for $j = 1,2 \ldots r$ and L^2-continuous and adapted.

We make the following assumptions on the functions involved in (2.5).
(H_2) for $x, y, x_k, y_k, z_k \in L^2(\ell), k = 1,2$ and $t \in [0,T]$ there exist constants $K_i^{(j)}$, $M_i^{(j)}$, $N_i^{(j)}$ and $K^{(j)}, M^{(j)}, N^{(j)}$, $i=1, 2, j=1,2\ldots,r$ such that

$$\| f_i^{(j)}(t, s, x) - f_i^{(j)}(t, s, y) \|_2 \leq K_i^{(j)} \| x-y \|_2,$$

$$\| g_i^{(j)}(t, s, x) - g_i^{(j)}(t, s, y) \|_2 \leq M_i^{(j)} \| x-y \|_2,$$

$$\| h_i^{(j)}(t, s, x) - h_i^{(j)}(t, s, y) \|_2 \leq N_i^{(j)} \| x-y \|_2,$$

for all $0 \leq s \leq t \leq T$ and

$$\|F^{(j)}(t, x_1, y_1, z_1) - F^{(j)}(t, x_2, y_2, z_2)\|_2 \leq K^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

$$\|G^{(j)}(t, x_1, y_1, z_1) - G^{(j)}(t, x_2, y_2, z_2)\|_2 \leq M^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

$$\|H^{(j)}(t, x_1, y_1, z_1) - H^{(j)}(t, x_2, y_2, z_2)\|_2 \leq N^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

for all $0 \leq t \leq T$.

Our next theorem gives sufficient conditions for existence and uniqueness of the solution of equation (2.5).

Theorem 2. : Suppose the hypothesis (H_2) holds. Then for any $x_0 \varepsilon L^2(\ell_{t_0})$ there exists a unique L^2- continuous adapted solution of the equation (2.5) on I with $x(t_0) = x_0$.

Remark : We note that Theorems 2.1 and 2.2 established in [2] are special cases of our Theorems 1 and 2 respectively. An interesting study relating to more general Itô type stochastic integrodifferential equations can be found in recent papers by Berger and Mizel [3] and Pachpatte [5].

The following theorems deal with the stability of the solutions of equations (1.1) and (2.5).

Theorem 3.: Assume that the hypothesis (H_1) holds. Let $y(t)$ and $z(t)$ for t

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$\geq t_o$ be the solutions of equation (1.1) with initial conditions $y(t_o) = x_0$ and $z(t_o) = x_o^*$ respectively with $x_0, x_0^* \in L^2(\ell_{to})$. For given $T > t_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x_0 - x^*_0\|_2 < \delta$ then $\| y(t) - z(t) \|_2 < \varepsilon$ for all $t \in [t_o T]$.

Theorem 4.: Let the hypothesis (H_2) hold. Suppose $y(t)$ and $z(t)$ for $t \geq t_0$ are the solutions of the stochastic integrodifferential equation (2.5) with initial conditions $y(t_0) = x_0$ and $z(t_0) = x_0^*$ respectively and $x_0, x_0^* \in L^2(\ell_{t0})$. For given $T > t_0$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $\| x_0 - x_0^* \|_2 < \delta$ then $\| y(t) - z(t) \|_2 < \epsilon$ for all $t \in [t_0, T]$.

Proofs of Theorems 1 and 2

Let T be fixed in I such that $t_0 \leq t \leq T$. We define

(3.1) $x^{(0)}(t) = x_0$,

and for $n \geq 0$,

$$(3.2)\quad x^{(n+1)}(t) = x_0 + \int_{t_0}^{t} F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s))\, ds + \int_{t_0}^{t} G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))\, dw(s) + \int_{t_0}^{t} dw(s) H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s)).$$

We first show that the process $x^{(n)}(t)$, $n \geq 1$ is well defined and it is adapted L^2- continuous on $[t_0 T]$. The functions $f_i(s, \tau, x_0)$, $g_i(s, \tau, x_0)$, $h_i(s, \tau, x_0)$, $i = 1, 2, F(s, x_0 A_1 x^{(0)}(s), A_2 x^{(0)}(s))$, $G(s, x_0 B_1 x^{(0)}(s), B_2 x^{(0)}(s))$ and $H(s, x_0 P_1 x^{(0)}(s), P_2 x^{(0)}(s))$ are adapted and L^2-continuous for $t_0 \leq \tau \leq s \leq T$. Thus it is easy to see that $x^{(1)}(t)$ is well defined for $t_0 \leq \tau \leq s \leq T$. Also we observe from the fact that continuity of a function implies boundedness of the function on compact intervals and using the isometry property we see that $t \to x^{(1)}$ is continuous map:

$$[t_0 T] \to L^2(\ell).$$

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Assume that $x^{(n)}(t)$ is adapted and L^2-continuous on $[t_0, T]$. Then, each of the functions $f_i(s, \tau, x^{(n)}(\tau))$, $g_i(s, \tau, x^{(n)}(\tau))$, $h_i(s, \tau, x^{(n)}(\tau))$, $i = 1, 2$, $F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s))$, $G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$ and $H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$ is adapted and L^2-continuous on $[t_0, T]$ and hence bounded. It can be easily seen that $x^{(n+1)}(t)$ is adapted and by using isometrics (2.2) and (2.3) we observe that $t \to x^{(n+1)}(t)$ is L^2-continuous on $[t_0, T]$. This proves that the process $x^{(n)}(t)$, $n \geq 1$ is well defined, adapted and L^2-continuous on $[t_o, T]$

We next consider the convergence of the sequence $\{x_n(t)\}$, $n \geq 0$ defined by (3.1)-(3.2). From (3.2) and the fact that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ we observe that

$$
\begin{aligned}
(3.3) \quad \| x^{(n+1)}(t) - x^{(n)}(t) \|_2^2 \leq\ & 3 \| \int_{t_0}^{t} [F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s)) \\
& - F(s, x^{(n-1)}(s), A_1 x^{(n-1)}(s), A_2 x^{(n-1)}(s))]\, ds \|_2^2 \\
& + 3 \| \int_{t_0}^{t} [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s)) \\
& - G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))]\, dw(s) \|_2^2 \\
& + 3 \| \int_{t_0}^{t} dw(s) [H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s)) \\
& - H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s))] \|_2^2 .
\end{aligned}
$$

From the isometries (2.2) and (2.3) the boundedness of the wave functions $v_1, \ldots, v_n$ over $[t_0, T]$, the hyposheris (H_1), $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and Schwarz inequality we observe that

$$
\begin{aligned}
(3.4) \quad \| \int_{t_0}^{t} & [F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s)) \\
& - F(s, x^{(n-1)}(s), A_1 x^{(n-1)}(s), A_2 x^{(n-1)}(s))]\, ds \|_2^2
\end{aligned}
$$

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$$\leq K^2 (t - t_0) \int_{t_0}^{t} [\| x^{(n)}(s) - x^{(n-1)}(s)\|_2$$

$$+ K_1 \int_{t_0}^{s} \| x^{(n)}(\tau) - x^{(n-1)}(\tau)\|_2 \, d\tau$$

$$+ (K_2^2 Q_1^2 \int_{t_0}^{s} \|x^{(n)}(\tau) - x^{(n-1)}(\tau)\|_2^2 \, d\tau)^{1/2}]^2 \, ds$$

$$\leq 3K^2(T - t_0) Q_1^* \int_{t_0}^{t} \| x^{(n)}(s) - x^{(n-1)}(s)\|_2^2 \, ds,$$

(3.5)
$$\| \int_{t_0}^{t} [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))] \, dw(s)\|_2^2$$

$$= \int_{t_0}^{t} \| [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))] \|_2^2 d\mu(s)$$

$$\leq C_1^2 \int_{t_0}^{t} \|G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s)) \|_2^2 ds$$

$$\leq 3C_1^2 M^2 Q_2^* \int_{t_0}^{t} \|x^{(n)}(s) - x^{(n-1)}(s)\|_2^2 ds,$$

and

(3.6)
$$\| \int_{t_0}^{t} dW(s) [H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

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$$- H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s))]\|_2^2$$

$$= \int_{t_0}^{t} \| H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

$$- H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s)) \|_2^2 d\mu(s)$$

$$\leq C_2^2 \int_{t_0}^{t} \|H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

$$- H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s)) \|_2^2 ds$$

$$\leq 3C_2^2 N^2 Q_3^* \int_{t_0}^{t} \|x^{(n)}(s) - x^{(n-1)}(s)\|_2^2 ds,$$

where

$$Q_1^* = \left[1 + \frac{K_1^2 (T-t_0)^2}{2} + Q_1^2 K_2^2 (T - t_0)\right]$$

$$Q_2^* = \left[1 + \frac{M_1^2 (T-t_0)^2}{2} + Q_2^2 M_2^2 (T-t_0)\right]$$

$$Q_3^* = \left[1 + \frac{N_1^2 (T-t_0)^2}{2} + Q_3^2 N_2^2 (T - t_0)\right]$$

and C_1, C_2, Q_1, Q_2 and Q_3 are constants depending on t_0 and T. Using (3.4) - (3.6) in (3.3) we get

$$(3.7) \quad \| x^{(n+1)}(t) - x^{(n)}(t) \|_2^2 \leq D \int_{t_0}^{t} \| x^{(n)}(s) - x^{(n-1)}(s) \|_2^2 ds,$$

where

$$(3.8) \quad D = 9 [K^2 (T - t_0) Q_1^* + C_1^2 M^2 Q_2^* + C_2^2 N^2 Q_3^*].$$

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By succesive iterations on (3.7) we have

$$\| x^{(n+1)}(t) - x^{(n)}(t) \|_2^2$$

$$\leq D^n \int_{t_0}^{t} \int_{t_0}^{s_{n-1}} \cdots \int_{t_0}^{s_1} \| x^{(1)}(s_{n-1}) - x^{(0)}(s_{n-1}) \|_2^2 \, ds \, ds_1 \cdots ds_{n-1}$$

We note that both $s \to x^{(1)}(s)$ and $s \to x^{(0)}(s) = x_0$ *are* L^2 - continuous and hence bounded on $[t_0, T]$. Thus by using Cauchy integral formula, we get

$$\| x^{(n+1)}(t) - x^{(n)}(t) \|_2^2 \leq D_1^n \frac{(t - t_0)^n}{n!}, \tag{3.9}$$

where D_1 is a positive constant depending on t_0 and T. From the inequality (3.9), for any $n > r$, it follows that

$$\| x^{(n+1)}(t) - x^{(r+1)}(t) \|_2 = \| \sum_{m=r+1}^{n} (x^{(m+1)}(t) - x^{(m)}(t) \|_2$$

$$\leq \sum_{m=r+1}^{n} \left[D_1^m \frac{(t - t_0)^m}{m!} \right]^{1/2}$$

Thus, we observe that there is an $x(t)$ in $L^2(\ell)$.such that $x^{(n)}(t)$ converges uniformly to $x(t)$ in $L^2(\ell)$ on $[t_0, T]$. Since each $x^{(n)}(t)$ is L^2 - continuous and adapted, the process $x(t)$ is L^2 - continuous and adapted on $[t_0, T]$.

Finally we show that the process $x(t)$, $t \geq t_0$ satisfies equation (1.1).. It is clear that $x(t_0) = x_0$. Also

$$\| \int_{t_0}^{t} [F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s))$$

$$- F(s, x(s), A_1 x(s), A_2 x(s))] \, ds \|_2^2 \tag{3.10}$$

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$$\leq 3K^2 (T - t_0)\, Q_1^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s)\|_2^2 ds$$

$$\to 0 \text{ as } n \to \infty$$

since, for $s \in [t_0, T]$, $x^{(n)}(s) \to x(s)$ uniformly in $L^2(\ell)$. Further we note that

(3.11) $$\| \int_{t_0}^{t} [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G(s, x(s), B_1 x(s), B_2 x(s))]\, dW(s)\|_2^2$$

$$\leq 3C_1^2 M^2 Q_2^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s)\|_2^2 ds$$

$$\to 0 \text{ as } n \to \infty.$$

and

(3.12) $$\| \int_{t_0}^{t} dW(s)[H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

$$- H(s, x(s), P_1 x(s), P_2 x(s))] \|_2^2$$

$$\leq 3C_2^2 N^2 Q_3^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s)\|_2^2 ds$$

$$\to 0 \text{ as } n \to \infty .$$

Thus from (3.2) and (3.10) - (3.12) it follows that

(3.13) $$x(t) = \lim_{n\to\infty} x^{(n+1)}(t)$$

$$= \lim_{n\to\infty} [x_0 + \int_{t_0}^{t} F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s))ds$$

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$$+ \int_{t_0}^{t} G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))\, dW(s)$$

$$+ \int_{t_0}^{t} dW(s)\, H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))\ ]$$

$$= x_0 + \int_{t_0}^{t} F(s, x(s), A_1 x(s), A_2 x(s))ds$$

$$+ \int_{t_0}^{t} G(s, x(s), B_1 x(s), B_2 x(s))\, dW(s)$$

$$+ \int_{t_0}^{t} dW(s)\, H(s, x(s), P_1 x(s), P_2 x(s))$$

This shows that $x(t)$, $t \geq t_0$ is a solution of the equation (1.1) with initial condition $x(t_0) = x_0$ for $t \in [t_0, T]$. Clearly $T > t_0$ is arbitrary, hence we can determine a solution $x(t)$ for all $t \geq t_0$.

We next prove the uniqueness of solutions of the equation (1.1). Let $x(t)$ and $y(t)$ be two L^2-continuous and adapted solutions of (1.1) such that $x(t_0) = y(t_0) = x_0$. By using $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, isometries (2.2) and (2.3), Schwartz inequality, hypothesis (H_1) and boundedness of $V_1, \ldots V_n$ on $[t_0, t]$ we observe that

$$(3.14)\quad \|x(t) - y(t)\|_2^2 \leq 3(t - t_0) \int_{t_0}^{t} \| F(s, x(s), A_1 x(s), A_2 x(s))$$

$$- F(s, y(s), A_1 y(s), A_2 y(s))\|_2^2\, ds$$

$$+ 3 \int_{t_0}^{t} \| G(s, x(s), B_1 x(s), B_2 x(s))$$

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$$- G(s, y(s), B_1 y(s), B_2 y(s))\|_2^2 \, d\mu(s)$$

$$+ 3\int_{t_0}^{t} \| H(s, x(s), P_1 x(s), P_2 x(s))$$

$$- H(s, y(s), P_1 y(s), P_2 y(s))\|_2^2 \, d\mu(s)$$

$$\leq D \int_{t_0}^{t} \| x(s) - y(s)\|_2^2 \, ds,$$

Where D is as defined in (3.8). Thus from Gronwall's inequality it follows that

(3.15) $$\|x(t) - y(t)\|_2^2 = 0, \quad t \in [t_0, T].$$

Since both $x(t)$ and $y(t)$ are L^2- continuous, from (3.15) we get

(3.16) $$\|x(t) - y(t)\|_2 = 0,$$

for all $t \in [t_0 T]$. Thus $x(t) = y(t)$, for each $t \in [t_0, T]$. This completes the proof of Theorem 1. The proof of Theorem 2 follows by arguments as in the proof of Theorem 1 with suitable modifications. We omit the details.

Proofs of Theorems 3 and 4

We define $y^{(n+1)}(t)$ and $z^{(n+1)}(t)$ for $n = 0, 1...$ with $y^{(0)}(t) = x_0$ and $z^{(0)}(t) = x_0^*$ similar to $x^{(n+1)}(t)$ with $x^{(0)}(t) = x_0$ defined in the proof of Theorem 1 for all $t \in I$. By using $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, the isometries (2.2) and (2.3), the Schwarz inequality, the boundedness of the wave functions $v_1, ..., v_n$ over $[t_0 T]$, the hypothesis (H_1) we observe that

(4.1) $$\| y^{(n+1)}(t) - z^{(n+1)}(t)\|_2^2 \leq 4 \|x_0 - x_0^*\|_2^2$$

$$+ 4D^* \int_{t_0}^{t} \| y^{(n)}(s) - z^{(n)}(s)\|_2^2 \, ds,$$

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where $D^* = D/3$ and D is as defined in (3.8). From (4.1) by successive iterations, we get

$$(4.2)\quad \| y^{(n+1)}(t) - z^{(n+1)}(t)\|_2^2 \le 4 \sum_{m=0}^{n} \frac{(4D^*)^m}{m!} (t - t_0)^m \|x_0 - x_0^*\|_2^2$$

$$+ (4D^*)^{n+1} \int_{t_0}^{t} \int_{t_0}^{s_1} \cdots \int_{t_0}^{s_n} \|y^{(0)}(s_{n+1}) - x^{(0)}(s_{n+1})\|_2^2 ds_{n+1} \cdots ds_1$$

$$\le 4 \sum_{m=0}^{n+1} \frac{(4D^*)^m}{m!} (t - t_0)^m \|x_0 - x_0^*\|_2^2 \le 4 \|x_v - x_0^*\|_2^2 e^{4D^*T},$$

for all t, $t_0 \le t \le T$ and the result follows. this completes the proof of Theorem 3. The proof of Theorem 4 follows by similar argument as in the proof of Theorem 3 with suitable modifications, and hence we omit the details.

Remark 2.

We note that, one can very easily establish convergence theorems for the solutions of equations (1.1) and (2.5) similar to the Theorem 5.2 given in [2] with suitable modifications. In concluding this paper we note that regarding the study of stochastic differential equations involving Itô-Clifford integrals many problems are still open. The authors have tried to obtain a formula similar to Itô formula for stochastic differential equations involving Itô-Clifford integrals but could not get satisfactory answer. Once such a formula is available it will give rise to the study of many problems relating to stochastic equations involving Itô-Clifford integrals. The study of boundary value problems and various properties of second and higher order stochastic differential equations involving Itô-Clifford integrals awaits for its development.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp- 47-50

REGULARLY PERIODIC ELEMENTS OF A GROUP RING

W. B. Vasantha Kandasamy*

(Received 13.9.91)

ABSTRACT

In this note the author continues the study of regularly periodic elements of a group ring. A necessary and sufficient condition for an element to be regularly periodic in a group ring is obtained. For more properties about regularly periodic elements of a ring please refer [1].

Definition 1

Let R be an arbitrary associative ring. An element x of R is said to be a Regularly periodic element of periodicity n, $(n \geq 2)$ if there exists n distinct elements in R different from x say $r_1, r_2, \ldots r_n$ $(r_1, r_2, \ldots r_n \in R)$ such that

$$x^2 = r_1 = x^{n+2} = x^{2n+2} = \ldots$$
$$x^3 = r_2 = x^{n+3} = x^{2n+3} = \ldots$$
$$x^4 = r_3 = x^{n+4} = x^{2n+4} = \ldots$$
$$\vdots$$
$$x^n = r_{n-1} = x^{2n} = x^{3n} = \ldots$$
$$x^{n+1} = r_n = x^{2n+1} = x^{3n+1} = \ldots$$

Throughout this paper K denotes a field and G a group: KG the group ring of G over K.

Proposition 1:

Let G be a finite group and K any field of characteristic zero. Then the group ring KG has regularly periodic elements.

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Proof.

Given G is finite, let $|G| = n$. *Consider* $\alpha = 1/n \Sigma g_i (g_i \varepsilon G)$ clearly $\alpha \in KG$ with $\alpha^2 = \alpha$. Hence KG has regularly periodic elements.

Proposition 2 :

Let G be any group having an element of finite order and K a field of characteristic zero. Then KG has a regularly periodic element. **Proof.** Let $g \varepsilon G$ be such that $g^m = 1\ (m < \infty)$. Consider $\alpha = \frac{1}{m} \sum_{i-0}^{m-1} g^i$ in KG. Clearly α is a periodic element of KG.

Proposition 3.

Let G be a torsion free abelian group and K any field; No nontrivial element of KG is regularly periodic.

Proof.

G is torsion free abelian hence KG is a domain. Therefore KG cannot contain even an idempotent. Thus KG has no nontrivial regularly periodic element.

Proposition 4.

Let G be a finite group and K any field. If $H_1, H_2, \ldots, H_n$ are subgroups of G, then the group ring KG has periodic elements.

Proof.

For each subgroup H_i of G. $\alpha = \frac{1}{|H_i|} \Sigma h_i \left(h_i \varepsilon H_i\right)$ is a regularly periodic element of KG. (H_i) denotes the number of elements on H_i).

Proposition 5.

Let $Z_2 = (0, 1)$ and S_n be the symmetric group of degree n. Then every element of $Z_2 S_n$ which is not nilpotent is regularly periodic.

Proof.

Since order of $Z_2 S_n$ is $2^{n!}$ $Z_2 S_n$ is a finite ring and hence every non nilpotent element is regularly periodic.

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Theorem 6.

Let G be an abelian group. The group ring KG has a nontrivial regularly periodic element if and only if G has an element of finite order.

Proof.

If G has an element of finite order by Proposition 2 the group ring KG has a regularly periodic element.

Conversely if KG has a regularly periodic element to prove G has an element of finite order. Let $\alpha = \Sigma \alpha_i g_i \varepsilon KG$ be regularly periodic; this implies there exists n distinct elements $\beta_1, \beta_2, \ldots\ldots\ldots \beta_n \varepsilon KG$ with

$$\alpha^2 = \beta_1 = \alpha^{n+2} = \alpha^{2n+2} = \ldots$$
$$\alpha^3 = \beta_2 = \alpha^{n+3} = \alpha^{2n+3} = \ldots$$
$$\alpha^4 = \beta_3 = \alpha^{n+4} = \alpha^{2n+4} = \ldots$$
$$\vdots$$
$$\alpha^n = \beta_{n-1} = \alpha^{2n} = \alpha^{3n} = \ldots$$
$$\alpha^{n+1} = \beta_n = \alpha^{2n+1} = \alpha^{3n+1} = \ldots$$

Clearly $\alpha^2 (\alpha^n - 1) = 0$ this implies $\alpha^n = 1$ or KG has divisors of zero. Now KG has no divisor of zero only when G is not torsion free. That is there are elements in G of finite order. $\alpha^n = 1$ is impossible if every element in G is torsion free thus G has elements of finite order. Hence the theorem.

Remark .

If we do not assume G is abelian we may not be able to conclude KG has no zero divisors if every element of G is torsion free. Thus the zero divisor conjecture for group rings can be reformulated as follows.

Conjecture.

Let G be a torsion free non abelian group and K any field. Can KG have regularly periodic elements?

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 51-54

FIXED POINTS FOR A PAIR OF DENSIFYING MAPPINGS

Rakesh Kumar Jain*

(Received 24 - 9- 1991)

ABSTRACT

In the present paper we establish a fixed point theorem for a pair of densifying mappings which includes the results or Khan and Fisher [2] and Furi and Vignoli [1].

Keywords and Phrases : Fixed point, densifying group,

Mathematics Subject classifications (1991) : 54 H25, 47 H10

INTRODUCTION

Kuratowski [3] introduced the concept of **measure of non-compactness** of a bounded set in a metric space as follows :

Defination 1 : Let A be a bounded set in a metric space (X,d). Then the **measure of non-compactness** of A, written as $\alpha(A)$, is the infimum of all $\varepsilon > 0$ such that A admits a finite covering by sets with diameter less than ε.

Definition 2 : [1] A mapping T of a metric space (X,d) into itself is said to be **densifying** if for every bounded subsets A of X with $\alpha(A) > 0$.

$\alpha(T(A)) < \alpha(A)$

Reich [4] obtained the following theorem.

Theorem A : Let T be a mapping of a compact metric space X into itself such that

$d(Tx, Ty) < (1/2)[d(x, Tx) + d(y, Ty)]$

for all $x, y \in X$ with $Tx \neq Ty$, then T has a unique fixed point.

MAIN RESULT

We present the following theorem.

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Theorem : Let S and T be continuous densifying mappings of a **complete metric space** (X,d) into itself such that for all x,y in X

$$F_1(Sx,TSy) < max\ \{[F_2(x,Sy),F_2(x,Sx),F_2(y,Sy),(1/2)F_2(x,Sx)$$

$$+ (1/2)F_1(Sy,TSy)]\ U\ min\ [F_2(x,TSy),\ F_2(Sy,Sx)]\}$$

and

$$F_2(Tx,STy) < max\{[F_1(x,Ty),F_1(x,Tx),F_1(y,Ty),(1/2)F_1(x,Tx)$$

$$+ (1/2)\ F_2(Ty,STy)]\ U\ min[F_1(x,STy),F_1(Ty,Tx)]\}$$

where F_1 and F_2 are real-valued functions of $X \times X$ into $[0, \infty)$ with either F_1 or F_2 lower semi-continuous. If for some x_o in X, the sequence $\{x_n\}$ defined by $Sx_{2n} = x_{2n+1}$, $Tx_{2x+1} = x_{2n+2}$ for $n = 0,1,2,.....$ is bounded and $F_1(x,x) = F_2(x,x) = 0$ for all $x \in X$, then either S or T has a fixed point z. Further, if z is a common fixed point of S and T, then it is the unique common fixed point of S and T.

Proof : We may assume that F_1 is lower semi-continuous. Consider the set $A = \{x_{2n+1} : n = 0,1,2, ...\}$. Then $ST(A) = \{x_{2n+1} : n = 1,2,.....\}$ and so $ST(A) \subset A$. The continuity of S and T now implies that $ST(\overline{A}) \subset \overline{ST(A)}$. Now suppose that $\alpha(A) > 0$. Then

$$\alpha(A) = \alpha(\{x_1\}\ U\ ST(A))$$

$$= max\ \{\alpha(x_1),\ \alpha(ST(A))\}$$

$$= \alpha(ST(A)) < \alpha(A)$$

giving a contradiction. This contradiction implies that $\alpha(A) = 0$ and so A is relatively compact. Since, the function F_1 is lower semi continuous, the function f defined by

$$f(x) = F_1(x,Tx)$$

for all x in $\overline{A}$ is lower semi-continuous and so assumes its minimum value at some z in $\overline{A}$. Thus if $Tz \neq z$ and $STz \neq Tz$ then $STz \in \overline{A}$ and

$$f(STz) = F_1(STz,\ TSTz)$$

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$< max \{[F_2(Tz, STz), F_2(Tz, STz), F_2(Tz, STz), (1/2) F_2(Tz, STz)$

$+ (1/2) F_2(STz, TSTz)] U min [F_2(Tz, TSTz), F_2(STz, STz)]\}$

$= max \{F_2(Tz, STz), (1/2) F_2(Tz, STz) + (1/2) F_1(STz, TSTz)\}$

We have either

$f(STz) < F_2(Tz, STz)$ or $f(STz) < (1/2) F_2(Tz, STz) + (1/2) F_1(STz, TSTz)$

which implies

$f(STz) < F_2(Tz, STz)$

Therefore

$f(STz) < F_2(Tz, STz)$

$< max \{[F_1(z,Tz), F_1(z,Tz), F_1(z,Tz), (1/2)F_1(z, Tz)$

$+(1/2)F_2(Tz, STz)] U min [F_1(z,STz), F_1(Tz,Tz)]\}$

$= max\{F_1(z,Tz), (1/2)F_1(z,Tz) +(1/2) F_2(Tz, STz)\}$

which implies

$f(STz) < F_1(z,Tz) = f(z)$

contradicting the definition of z. It follows that either $Tz = z$, in which case z is a fixed point of T, or $STz = Tz$, in which case $z' = Tz$ is a fixed point of S.

Now suppose that z is a common fixed fixed point of S and T. Then if S and T have a second distinct common fixed point z' then $z' \neq Sz'$, $z \neq Tz'$, and so.

$F_1(z,z') = F_1(Sz, TSz')$

$< max \{[F_2(z,Sz'), F_2(z,Sz), F_2(z',Sz'), (1/2) F_2(z,Sz)$

$+(1/2) F_1(Sz', TSz')] U min [F_2(z,TSz'), F_2(Sz',Sz)]\}$

$= F_2(z,Sz')$

$= F_2(Tz, STz')$

$< max \{[F_1(z,Tz'), F_1(z,Tz), F_1(z', Tz'), (1/2) F_1(z,Tz)$

$+ (1/2) F_2(Tz',STz')] U min [F_1(z,STz'), F_1(Tz',Tz)]\}$

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$= F_1(z,z')$

giving a contradiction. The common fixed fixed point z must therefore be unique. This completes the proof of the theorem.

Corollary 1 [2] : Let S and T be continuous densifying mappings of a complete metric space (X,d) into itself such that for all x,y in X

$F(Sx, TSy) < F(x,Sy), x \neq Sy$

and

$F(Tx, STy) < F(x,Ty), x \neq Ty,$

where F is a real valued lower semi-continuous function fo XxX into $[o, \infty]$. If for some x_0 in X, the sequence $\{x_n\}$ defined by $Sx_{2n} = x_{2n+1}$, $Tx_{2n+1} = x_{2n+2}$ for $n = 0,1,2,....$ is bounded, then either S or T has a fixed point z. Further, if z is a common fixed point of S and T, then it is the unique fixed point of S and T.

Corollary 2 [1] : Let T be a continuous densifying mapping of a complete metric space (X,d) into itself and let F, be a real valued lower semi-continuous function of XxX into $[0, \infty]$ such that for all x,y in X.

$F(Tx,Ty) < F(x,y), x \neq y$

If for some x_0 in X the sequence $\{x_n\}$ defined by $Tx_{2n} = x_{2n+1}$ for $n = 0,1,2,.....$ is bounded, then T has a unique fixed point z.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp-55-58

AN ALTERNATIVE ESTIMATOR FOR THE MEAN OF FINITE POPULATION USING AUXILIARY VARIABLES

Gulab Singh*

(Received: 30-03-1991, after revision: 23-10-1991)

ABSTRACT

The supplementary information on auxiliary variables is very useful at the estimation stage in sample surveys. Several estimators developed using the supplementary information on the auxiliary variables are available in the literature. In this paper an alternative estimator of the population mean ($\overline{Y}$) using supplementary information on two auxiliary variables x (positively correlated with y) and z (negatively correlated with y) has been suggested. In section 2 the bias and the mean squared - error (MSE) of the proposed estimator have been obtained and in section 3 efficiency of the proposed estimator has been compared with the usual unbiased estimator $\overline{y}$ and with that of Srivastava [3] under simple random sampling with replacement (SRSWR).

Mathematics Subject Classification (1991) : 62 D 05

Keywords: Auxiliary variable, mean square error, ratio estimator, simple random sampling with replacement (SRSWR), unbiased estimator

INTRODUCTION

In large sample surveys, data on more than one auxiliary variables are often collected. Some of these variables may be correlated with the main character under study (y). Several successful attempts have been made to use multi-auxiliary information in increasing the precision of estimator of the population mean $\overline{Y}$. Olkin [1] extended the ratio method of estimation to incorporate supplementary information on several auxiliary variates each positively correlated with y. The case of multi-auxiliary variates, some positively correlated with y and the others negatively correlated with it, has been discussed by Srivastava [3,4] and Rao and Mudholkar [2].

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Let a simple random sample of size n is drawn from a population of size N. Let $\bar{Y}$, $\bar{X}$ and $\bar{Z}$ be the population and $\bar{y}$, $\bar{x}$ and $\bar{z}$ be the sample means of the characters y, x (positively correlated with y) and z (negatively correlated with y) respectively. Two estimators of the population mean $\bar{Y}$ based on the auxiliary information are well known in the literature, namely, the ratio estimator $\bar{y}(\bar{X}/\bar{x})$ and the product estimator $\bar{y}$ $(\bar{z}/\bar{Z})$. Srivastava [3], however, suggested an estimator for $\bar{Y}$ using the supplementary information on both the variables x and z, as

$$\bar{y}_{Rw} = \bar{y}\ [w_1(\bar{X}/\bar{x}) + w_2(\bar{z}/\bar{Z})] \tag{1}$$

where w_1 and w_2 are the weights associated with the two estimators such that $w_1 + w_2 = 1$ and studied its properties.

Further let $\bar{x}^*$ and $\bar{z}^*$ be the means corresponding to the $(N - n)$ units no included in the sample. An alternative estimator ($\bar{y}_{Rw'}$) of $\bar{Y}$ associated with x and z can be constructed which is linear in $(u/U)^{\delta u}$ where u represents x, z; U represents X, Y and $\delta_u = sign\ (\rho_{yu})$. In other words.

$$\bar{y}_{Rw'} = \bar{y}[\ w'_1(\bar{x}^*/\bar{X}) + w'_2(\bar{Z}/\bar{z}^*\ )] \tag{2}$$

Where w_1 and w_2 are the weights such that $w'_1 + w'_2 = 1$

BIAS AND MEAN SQUARED ERROR OF $\bar{y}_{Rw'}$

Following Sukhatme and Sukhatme [5], the bias and the MSE of the estimator $\bar{y}_{Rw'}$ to the order of $O(n^{-1})$ can easily be seen to be,

$$B(\bar{y}_{Rw'}) = g\bar{Y}n^{-1}[w'_2(g\,C_z^2 + C_{zy}) - w'_1C_{xy}] \tag{3}$$

and

$$MSE(\bar{y}_{Rw'}) = \bar{Y}^2n^{-1}[w_1'^2(g^2C_x^2 + C_y^2 - 4gC_{xy}) + w'^2_2(3g^2C_z^2 + C_y^2 + 4gC_{zy}) + 2w_1'w_2'(g^2C_x^2 - g^2C_{xz}^2 + C_y^2 + 2gC_{zy} - 2gC_{xy}) - 2w'_2(g^2C^2_z + gC_{zy}) + 2w_1'\,g\,C_{xy}] \tag{4}$$

where,

$g = n/(N-n)$, $C_x^2 = \sigma_x^2/\bar{X}^2$, $C_y^2 = \sigma_y^2/\bar{Y}^2$, $C_z^2 = \sigma_z^2/\bar{Z}^2$,
$C_{xy} = \rho_{xy}C_xC_y$, $C_{zy} = \rho_{xy}C_zC_y$, $C_{xz} = \rho_{xz}C_xC_z$,

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$\rho_{xy} = Cov(x,y)/\sigma_x\sigma_y$, $\rho_{yz} = Cov(y,z)/\sigma_y\sigma_z$, $\rho_{xz} = Cov(x, z)/\sigma_x\sigma_z$

The optimum weights obtained by minimizing (4) subject to the condition that $w'_1 + w'_2 = 1$ can be seen to be

$$w'_1 = \frac{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}}{g(C_x^2 + C_z^2 + 2C_{xz})} \quad \text{and}$$

$$w'_2 = \frac{g(C_x^2 + C_{xz}) + C_{xy} + C_{yz}}{g(C_x^2 + C_z^2 + 2C_{xz})} \tag{5}$$

The optimum values of bias and MSE can be obtained by substituting the values of w'_1 and w'_2 in equations (3) and (4) as,

$$B_0(\bar{y}_{Rw'}) = \frac{g\bar{Y}}{n}\left[\left(gC_z^2 + C_{zy}\right) - \frac{\{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}\}(gC_z^2 + C_{zy} + C_{xy})}{g(C_x^2 + C_z^2 + 2C_{xz})}\right] \tag{6}$$

and

$$M_0(\bar{y}_{Rw'}) = \frac{\bar{Y}^2}{n}\left[\left(g^2C_z^2 + C_y^2 + 2gC_{zy}\right) - \frac{\{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}\}^2}{g(C_x^2 + C_z^2 + 2C_{xz})}\right] \tag{7}$$

In practice, the population coefficients of variation (C_y, C_x, C_z) and the correlation coefficients (ρ_{xy}, ρ_{zy} and ρ_{xz}) are not known so that optimum weights will have to be estimated from the sample values. However, what may be known in practice may be the nature of the correlation between the variables y, x and z. This knowledge is essential for constructing the estimator. The sample estimates of optimum weights can be obtained by replacing each term in (5) by their sample estimates.

For the sake of simplicity, we may assume that

$$C_x = C_z \text{ and } \rho_{yx} = -\rho_{yz} \tag{8}$$

so that

$$w'_1 = w'_2 = \tfrac{1}{2} \tag{9}$$

From (6), (7), (8) and (9) the optimum bias and MSE of $\bar{y}_{Rw'}$ to the order of $O(n^{-1})$ can be found to be,

$$B^*_0(\bar{y}_{Rw'}) = g\bar{Y}(2n)^{-1} C_x(gC_x - 2\rho_{xy}C_y) \tag{10}$$

and

$$M_0^*(\bar{y}_{Rw'}) = \bar{Y}^2 n^{-1}[C_y^2 + (g^2C_x^2/2)(1-\rho_{xz}) - 2gC_xC_y\rho_{xy}] \tag{11}$$

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The estimator $\bar{Y}_{Rw'}$ is unbiased when $g = 2\rho_{yx}C_y/C_x$

It may be noted that when $w_1 = w_2 = 1/2$, Srivastava [3] obtained the minimum MSE of the estimator $\bar{y}_{Rw'}$ as,

$$M_0(\bar{y}_{Rw}) = \bar{Y}^2/n^{-1}\,[C_y^2 + (C_x^2/2)(1-\rho_{xz}) - 2\rho_{xy}\,C_xC_y] \qquad (12)$$

EFFICIENCY COMPARISONS

The variance of the usual unbiased estimator $\bar{y}$ is given by

$$V(\bar{y}) = \bar{Y}^2C_y^2 / n$$

From (11), (12), and (13) it can easily be shown that $\bar{y}_{Rw'}$ will be more efficient than $\bar{y}$ and $\bar{y}_{Rw}$ when

$$g/4 \le \rho_{yx}\,C_y / \{(1-\rho_{xz})\,C_x\} \le (1+g)/4$$

Which is true when $(1-g) \ge 0$, that is when $n \le N/2$. This condition is quite obvious as the proposed estimator is dual of the estimator proposed by Srivastava [3].

ACKNOWLEDGEMENT

The author is thankful to the referee for his constructive comments on the earlier version of the paper

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 59-66

RELIGIOSITY AND LOCUS OF CONTROL AMONG COLLEGE STUDENTS

O.P. Misra* & S.K. Srivastava*

(Received 25-11-1991)

ABSTRACT

Present investigation attemptes to study religiosity and locus of control among college students. The sample comprises of 200 college students (100 males and 100 females)po selected from Hardwar District by incidental sampling. Results reveal that female are more religious than male students, and sex differences exist in regard to locus of control among college students.

Mathematical Subject classifications (1991) : 92 J 99

Keywords & phrases : Religiosity,

INTRODUCTION

Religion is a powerful institution which plays an important role in shaping the social behaviour. In the nineteenth and twentieth centuries numerous ideas have been put forward regarding psychological and sociological factors that are responsible for the origin of religion.

The most important of these are (i) the Marxian theory which maintains that religion is one of the ideological reflection of the current state of economic interactions in a society, (ii) the similar, but more elaborately developed theory of the sociologist Emile Durkheim holds that religious beliefs constitute a projection of the structure of society, and (iii) the Freudian theory which emphasises that religious beliefs arise from projections designed to alleviate certain kinds of unconscious conflicts (7).

Galloway (8) has defined religion as the faith in a power beyond himself whereby he seeks to satisfy emotional needs and gain stability of life, which he expresses in the act of worship and service. The real religion is equally acceptable to all minds, and is almost equally philosophic, emotional, mistic and conducive to action. Williams and Cole (31) found in their study that

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religious persons carry less feeling of insecurity than non-religious persons. Mayo, et al. [16] concluded on the basis of their study that the religious men are significantly less depressed, less schizophrenic and less psychopathic deviant than non-religious ones. Adjustment and social responsibility were found to be possible correlates of religiousness (cf. [32]). Tiwari, Singh and Srivastava [28] found that high religious value oriented persons have better pattern of adjustment than low religious value oriented persons. In a study, Bhushan [3] found that religious girl students were significantly better adjusted than non-religious girl students.

Hassan and Khalique [10] showed that muslims tended to have high degree of religiousity in comparision to Hindus. Status (high/low), and sex (male/female) did not influence the religious attitudes of either Hindus or Muslims. Religiosity was found to have significant positive correlations with anxiety, rigidity and intolerance of ambiguity. Verma and Upadhyay [30] found that high religious students showed less social distance with others, if ethnic groups were treated as a whole. However, the students in general did show quite different attitudes towards various ethnic groups, and this is equally true with low and high religiosity groups, when treated separately. Sinha and Jha [27] concluded that correlation between religiosity and the total as well as the area-wise scores of religious prejudices revealed no significant relationship between the two, and high and low religous persons did not differ significantly on their religious prejudice scores.

The concept of locus of control was developed by Phares [19] in relation to beliefs regarding internal versus external control of reinforcement. It is assumed that individuals develop a general expectancy regarding their ability to control their lives. People who believe that the events that occur in their lives are a result of their own behaviour and/or personality characteristics are said to have an "expectancy of internal control", while people who believe events in their lives to be a function of luck, chance, fate, powerful others or powerful beyond their control or comprehension are said to have an "expectancy of external control".

Rotter [22] provides a useful tool for measuring individual differences to the extent to which reinforcement is viewed as a consequence of one's own behaviour or a consequence of such forces as chance, fate or powerful others. Singh [24] found that socio-culturally deprived students were significantly more external and chance oriented as compared to non-deprived

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students. Significant differences between internal - external locus of control in group conformity behaviour were found by Singh [25]. The results of the study conducted by Rao, et al. [20] suggested that the relationship between locus of control orientation and adjustment was not very clear. Locus of control orientation was found to determine the use of specific coping behaviours, but did not play a significant role in determining the experience and perception of stressful life events. Several studies have been carried out on religiosity, locus of control and other related variables (see, for instance, [1], [4], [9], [12], [14], [15], [23], [29] and [33]. The present paper reports data pertaining to the religiosity and locus of control among male and female college students.

Hypotheses :

(i) There would be significant difference between male and female students in terms of religiosity.
(ii) There would be significant difference between male and female students in terms of locus of control.

Sample :

The sample comprises of 100 male and 100 female students studying in class XI and XII in Hardwar District. The sample was selected by incidental sampling technique.

Scales :

Bhushan's Religiosity scale [2] and Rotter's Locus of control scale as standardized by Kumar and Srivastava [13] were used to measure the two variables of religiosity and locus of control among college students.

RESULTS AND DISCUSSION

The results confirm the contention of the hypothesis of significant difference between male and female students on religiosity scores (Table 1). The female students with a mean of 128.39 are found to be more religious in comparison to male students with a mean score of 121.62. This difference may be attributed possibly to cultural training and child rearing practices in the society. These appears to be a different role expectations in regard to boys and girls. Besides the temperamental difference also attracts girls towards religion. Similar conclusions may be found in the investigations

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conducted by Helode and Barlinge [11], and Bhushan and Singh [4].

Table - 1

Showing Mean and Standard Deviatioin of Male and Female Students in Terms of Religiosity

Students	N	Religiosity Scores		
		Mean	S.D.	t-value
Male	100	121.62	17.34	2.82
Female	100	128.39	16.58	P < 0.01

Table - 2

Showing Mean and Standard Deviatioin of Male and Female Students in Terms of Locus of Control.

Students	N	Locus of control scores		
		Mean	S.D.	t-value
Male	100	9.96	3.13	3.40
Female	100	11.56	3.43	P < 0.01

The results in regard to locus of control confirm that the female students are more external in their locus of control orientation than boys (Table 2). Female students are found to have developed a high "expectancy of external control". They believe that the events in their lives are controlled by such factors as luck, chance and super natural powers which are beyond there control. On the other hand the male students have greater faith in their own abilities and believe that their own abilities and efforts determine and shape their future.

The genesis of this difference can be directly traced in the amount of faith which boys and girls have in religion as discussed earlier, religion is a "faith in a power beyond himself; Since girls are more religious they have greater faith in super natural powers.

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These findings are in line with other studies reported by Person and Schneider [18], Blass [5] and Chandran [6] and Kunhikrishnan and Mathew [12] on sex differences in Internal-External locus of control. Results of Helode and Barlinge [11] indicate that externality and religiosity are positively correlated. Females are more religious than males. Molinari and Khanna [17] have suggested that adopting an external orientation when situations are uncontrollable as a defense against negative self-evaluation is a positive characteristic. Such persons attribute failure to luck as a defensive manoeuvre, and yet, resemble internals on behavioural indices of task performance.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 67-74

ON THE SOLUTIONS OF VISCOUS INCOMPRESSIBLE FLUID FLOW

N. Yadav*

(Received 13-12-1991)

ABSTRACT

A technique has been developed to combine two or more known solutions of uniform flow of viscous - incompressible fluid in pipes to a single new solution. Also in the light of the descussion a very bold proposition in the form of a principle has been put forward. This principle has been named as ***the principle of second degree harmonics***. The complete study has been divided into two sections. Section I deals with the formation of a new technique to combine two or more known solutions of viscous - incompressible fluid - flow to a single one and its applications. Section II deals with a particular case, how to derive a solution for a regular figure of n-sides from the solution for an equilateral triangle.

Mathematics Subject Classifications (1991): 76

Keywords and Phrases: Compoundable, viscous - incompressible fluid, Boussincsq's' formula.

SECTION I

In a problem of uniform flow of viscous - incompressible fluid in pipes, the boundary along which the velocity vanishes is a closed figure. Any section of this closed figure by a chord, we shall call a segment. Every chord divides the figure into two segments.

Let us consider two such segments taken from two different solutions and they are named as A and B, the bounding chord being PQ and RS (fig. 1).

We shall enquire into necessary and sufficient conditions under which A and B may be regarded as two segments of the same pipe so that by combining the two segments we may arrive at a new solution. The conditions must be :

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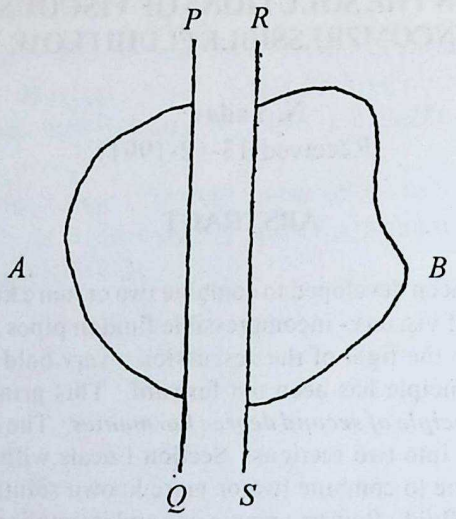

Fig. 1

Geometrical Ccondition :

Chord PQ must be equal to RS

Mechanical Ccondition :

The velocity function in A at every point of PQ must have the same value as the velocity function in B at the corresponding point in RS. By correspondence we mean the coincidence in superposition of PQ over RS.

The mechanical condition guarantees that along the planes of PQ and RS the flow will be the same at every point, so the flow on either side of this common plane will not be disturbed.

Two segments satisfying the geometrical and mechanical conditions, we shall call ***compoundable***.

Segments of two known solutions may be joined to give a new solution if they are compoundable. A segment is always compoundable to its mirror image for if D is the mirror image of C as shown in Fig. (2), then obviously the geometrical and mechanical conditions are satisfied.

Now we proceed to consider instead of segments the sector bounded by

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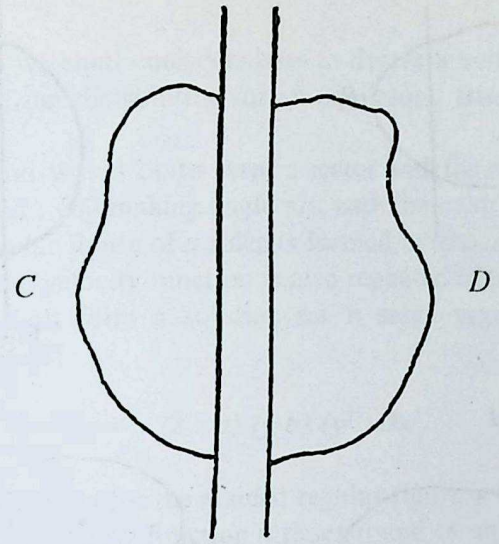

Fig. 2

radii vector making an angle $2\pi/n$, n being an integer. If n sectors be placed in such a way that each is the mirror image of adjoining one, the vertices coinciding, then the geometrical and mechanical conditions for compoundability are satisfied. Only if n is odd the two radii vector must be of equal length. When n is even they may be of unequal length.

We shall now consider some applications of the technique developed above. Considering the segments of an ellipse. Let A be the major segment containing the point of maximum velocity and B, a minor segment (Fig. 3). By combining each with its mirror image we arrive at two solutions L and M (Fig. 4 & 5) respectively. M is specially notable for its peculiarity of having two points of maximum velocity. By taking a major segment of M as indicated by dotted lines, we may get a solution with four points of maximum velocity as shown in Fig. 6. There is no limit to the variety of solutions that can be derived from one known solution. Next, let us consider a sector bounded by radii vectors making angle $2\pi/n$ n being an integer. Taking again the ellipse $PLQN$, let us form two sectors, one major sector containing the centre 0, and the other a minor sector, by two equal intersecting chords PQ and LN, intersecting at a point M on the major axis (Fig. 7).

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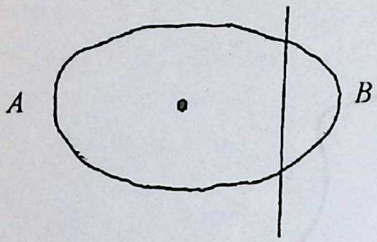

Fig. 3

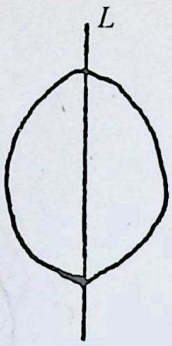

Fig. 4

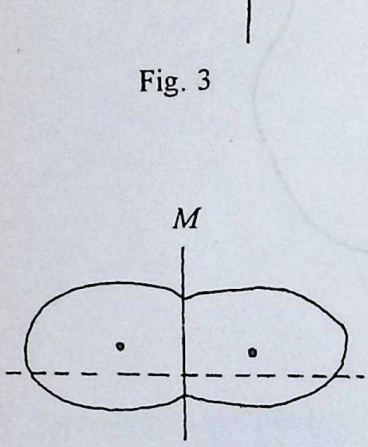

Fig. 5

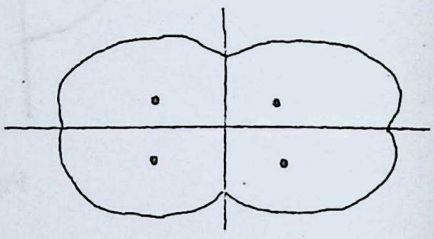

Fig. 6

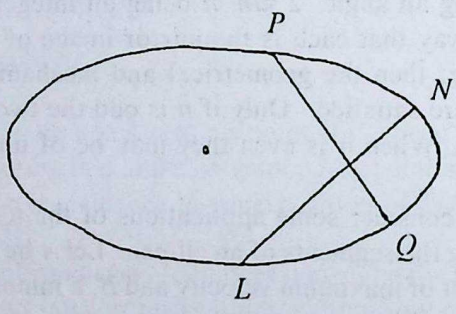

Fig. 7

Fig. 3, 4, 5, 6 & 7

The minor sector repeated n times gives a solution when the boundary consists of n arcs of an ellipse. In this solution there is one point of maximum velocity. The major sector repeated n times is more remarkable as it gives a solution with n points of maximum velocity.

In all such cases the area, the total flow per unit time, average velocity, maximum velocity, k and k', are calculated by considering a single segment or sector and multiplying by the necessary integer.

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SECTION II

In this section we shall consider, how to derive a solution for regular figure of n sides, from the solution for an equilateral triangle.

The best method would be to form a sector with the centre O as vertex and radii vectors OP, OQ making angle π/n with the central axis as shown in Fig. 1. The regular figure of n sides is formed by repeating the sector n times round O. The velocity function is also repeated in the same manner. In this way we shall form a solution for n sided regular figure from Boussinesq's formula,

$$w = - (K/4b) (y-b) (y^2 - 3x^2)$$

The velocity function for the n sided regular figure will be obtained by repeating the above velocity function n times round O' as shown in Fig. 2.

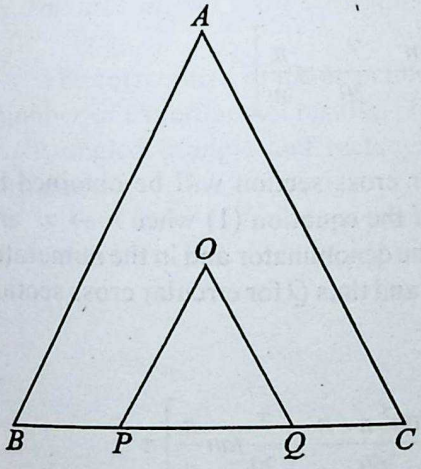

Fig. 1

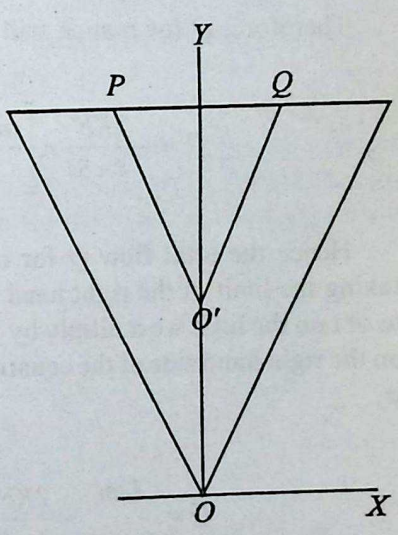

Fig. 2

Fig. 1 & 2

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We shall calculate the total flow in the sector $PO'Q$ and thus get the total flow for the regular n sided figure.

For the sector $PO'Q$, total flow Q is given by,

$$2\int_{\theta=0}^{\theta=\frac{\pi}{n}}\int_{r=0}^{r=\frac{b}{3\cos\theta\, k}} -\left(K/4b\right)\left[r^3\cos^3\theta - 3r^3\cos\theta\sin^2\theta + br^2\cos^2\theta + br^2\cos^2\theta + br^2\sin^2\theta - 4b^3/27\right] rdrd\theta$$

$$2\int_0^{\pi/n} -\frac{Kb^4}{4\times 81}\left[\frac{1}{20\cos^4\theta} - \frac{2}{5\cos^2\theta}\right]d\theta$$

$$Q = -\frac{Kb^4}{4\times 81}\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]$$

Therefore, Q for n such will be given by

$$Q = -\frac{2Kb^4}{4\times 81}n\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]$$

Hence the total flow Q for circular cross section will be obtained by taking the limit of the right hand side of the equation (1) when $n \to \infty$ and to obtain the limit we multiply by π in the denominator and in the numerator on the right hand side of the equation (1) and thus Q for circular cross section is,

$$Q = \lim_{n\to\infty} -\frac{2Kb^4}{4\times 81}\frac{\pi}{n}\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]\pi$$

$$Q = \frac{2Kb^4}{4\times 81}\cdot\frac{7}{20}\pi = \frac{kb^4}{\frac{40}{7}}\pi$$

Since $a = b/3$; a will be the radius of the circular cross section. We thus

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see that the total flow in tube of circular cross section derived from *Boussinesq's* formula for equilateral triangle does not come equal to the usual value, but is *greater* in the ratio 7:5.

It has been established by the author in his paper' *A generalized study of a viscous incompressible fluid flow through various cross sections of a tube*' that the formula for the total flow in circular tube, when derived from the formula for rectilinear figure, agrees with the usual value. One thing of remarkable importance is that both the usual velocity function for circular cross section and our formula for rectilinear figures, contain harmonics of second degree only. *Boussinesq's* formula, however, contains harmonics of third degree.

There does not seem to be any flaw in the discussion at section (I), so we are tempted to put forward a very bold proposition in the form of a principle. This principle we shall call the *principle of second degree harmonics*. The principle may be stated in the following words.

In all cases of Laminar flow of viscous incompressible fluids through tubes of cross section of any shape the velocity function can not contain Harmonics higher than those of second degree.

The correctness of the principle has yet to be established from sufficient number of experimental results. Thus the solutions for equilateral triangle, right angled triangle and rectangles can be modified in the light of this principle.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 75-82

VEDIC GEOMETRY*

S. L. Singh*

(Received 23-12-1991)

ABSTRACT

This talk intends to discuss *Śulbas'* period, and give some glimpses from *Brāhmaṇas* and *Śulbas'* concerning the origin of Pythagorian triplets and the irrationality concept of real numbers.

Mathematics Subject Classifications (1991) : O1A32

Keywords and Phrases: *Śulba Sūtra*, *trita*, pythagorian triplets

INTRODUCTION

According to the *Jaina* canon, geometry is the lotus of Mathematics (see *Sūtrakrta* (II. 1. 154), cf. (10), p. XII). Ancient Indian savants are now credited for fundamental pioneering work in geometry A very deep analysis by Seidenberg (See [21], p. 108) concludes "... Greek geometry (especially the Theorem of Pythagoras) did not somehow make its way into *Vedic* geometry, as Greek geometry is only supposed to have started about 600 B.C.".

First I quote the following *mantras* from *Yajurveda* (see [17], p. 210):

एका॑ च मे ति॒स्रश्च॑ मे ति॒स्रश्च॑ मे पञ्च॑ च मे॒ पञ्च॑ च म स॒प्त च मे स॒प्त च मे॒ नव॑ च मे॒ नव॑ च मऽएका॑दश च मऽएका॑दश च मे॒ त्रयो॑दश च मे॒ त्रयो॑दश च मे॒ पञ्चद॑श च मे॒ पञ्च॑दश च मे स॒प्तद॑श च मे स॒प्तद॑श च मे॒ नव॑दश च मे॒ नव॑दश च मऽ एक॑वि॒ँशति॑श्च मऽ में त्रयो॑वि॒ँशति॑श्च मे॒ त्रयो॑वि॒ँशति॑श्च मे॒ पञ्च॑वि॒ँशति॑श्च मे॒ पञ्च॑वि॒ँशति॑श्च मे स॒प्तवि॒ँशति॑श्च मे स॒प्तवि॒ँशति॑श्च मे॒ नव॑वि॒ँशति॑श्च मे॒ नव॑वि॒ँशति॑श्च मऽएकत्रि॒ँशच्च मऽएकत्रि॒ँशच्च मे॒ त्रयस्त्रि॑ँशच्च मे य॒ज्ञेन॑ कल्पन्ताम् ॥१ ॥

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*Talk presented at National Seminar on Science in Ancient India, Kumaun Univ., Nainital, Oct. 12-16, 1991.

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च त॑ स्रश्च मे॒ऽष्टौ च मे॒ ऽष्टौ च॑मे॒ द्वादश च मे॒ द्वाद॑श च मे॒ षोड॑श च मे॒ षोड॑श च मे॒ वि॒ँशतिश्च॑ मे॒ वि॒ँशतिश्च मे॒ चतु॑र्वि॒ँशतिश्च मे॒ चतु॑र्वि॒ँशतिश्च मे॒ऽष्टावि॒ँशतिश्च मे॒ द्वात्रि॒ँशच्च मे॒ द्वात्रि॒ँशच्च षट्त्रि॒ँशच्च मे॒ षट्त्रि॒ँशच्च मे॒ चत्वारि॒ँशच्च मे॒ चत्वारि॒ँशच्च मे॒ .चतु॑श्चत्वारि॒ँशच्च मे॒ चतु॑श्चत्वारि॒ँशच्च मेऽष्टाच॑त्वारि॒ँशच्च मे॒ यशे॑न कल्पन्ताम् ॥२ ॥

य॰ अ॰ १८ । मं॰ २४,२५ ॥

The first *mantra* means 1, 3, 5, 7, etc. It also means 1, 3^2, 5^2, 7^2, ...The second *mantra* means 4, 8, 12, 16 etc. For several such interpretations, refer to [3]. However, *Svāmī Dayānand* [17] says that several techniques of arithmetic may be derived from these *mantras*, and similarly fundamentals of algebra and geometry may be obtained from *Vedas*.

The following *mantra* from *Rig Veda* (see [17], p. 212) or [3, p. 326] talks about measuring instrument, circumference etc. :

कासी॑त् प्र॒मा प्र॑तिमा किं नि॒दानमाज्यं॑ किमा॑सीत् परि॒धिः क आ॑सीत् ।
छन्दः॒ किमा॑सी॒त् प्रउ॑गं॒ किमु॒क्थं यद्दे॒वा दे॒वमय॑जन्त॒ विश्वे॑ ॥४ ॥

ऋ॰ अ॰ ८ । अ॰ ७ । व॰ १८ । मं॰ ३ ॥

References of circle and its circumference, triangle and polygons are available in *Rig Veda* (see [3], pages 326-327).

TRITAH

Surprisingly Sanskrit name of π is *Trita*. In *Atharva Veda*, the ratio of circumference of a circle to its diameter is called *trita*, which is approximately 3:1 (see [3], p. 326).

According to Wendy O'Flaherty : "The riddles in the *Rig Veda* ... do not have, nor are they meant to have, answers. They are not merely rhetorical, but are designed to present one half of a Socratic dialogue through which the reader becomes aware of the inadequacy of his certain knowledge". I wish to say that neither it is, nor it can be the intention of the *mantra* to give a good approximation of *trita*. It is enough that the intended meaning of *trita* is not a rational number implicitly. There are several references in *Mahābhārata* which imply that the *trita* is 3. See [4] for a detailed analysis of *trita* or π in *Mhābhārata*, and [6] for a detailed analysis of the value of *trita* or π available in ancient Indian mathematical compositions. It may be mentioned that *Āryabhata's Āryabhatiya* (499 A.D.)., which is the earliest mathematical text that bears the name of the author, gives a good approximation of π (see also [1], [2], [7], [9], [10], [12], [23] - [25]).

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VEDIC PERIOD

A section of *Vaidikas* hold *Veda* as beginningless (indeed as old as this creation, (see [17], pp. 12-37) and *Svayambhu*. However, the *savites* hold that *Vāgisā Siva* set forth the *Vedas*, the six *aṅgas* and *āgamas*. According to seventh century Tamil *Saivite saint Appar, Lord Siva* is the embodiment of the four *Vedas* and six *aṅgas* (see, for instance, [13], Chap. 6). However, following Keith, and Satya Prakash (see [15], pp. 5-6; and also Renfrew [16] and Datta Singh [2], Part I, pp. 1-2), *Vedas*, the earliest extant compositions, and the subsequent *Vedic* literature are assigned the following dates :

Vedas	3000 B.C.
Brāhmaṇas	2500 - 2000 B.C.
Baudhāyana Sūtra	800 B.C.
Āpastambha Sūtra	600 B.C.
Manava Sūtra	550 B.C.
Pānini Sūtra	500 B.C.
Kātyāyana Sulba Sūtra	200 B.C.

Two recent investigations push back the above mentioned date of *Vedas* significantly. The recent deciphering of Harappan language by Louisiana State University Professor and Cryptologist, S. Kak suggests *Vedic* period around 8000 B.C. A recent deep analysis under the title "*Mahayuga*: the great cosmic cycle and the date of the *Veda*" concludes the *Ṛg Veda* is at least as old as 7300 B.C. If we accept 8000 B.C. for *Ṛg Veda*, the other dates of *Brāhmaṇas* and *Sūlba Sūtras* should also be pushed back proportionately.

THEOREM ON RECTANGLES

Pythagoras theorem (also called *Baudhāyana - Pythagoras* theorem in some recent High School text books) is available implicitly in *Brāhmanas* and explicitly in *Śūlba Sūtras* which mewans "*Sūtras of cord*". For a detailed analysis refer to Seidenberg [20] - [21] (see also [2], [11], [12], [14], [15], [24] and [25]).

Consider two sets of *mantras* from *Śatapatha Brāhmaṇa*.

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Set I :

तद्य एष पूर्वार्ध्यो वर्षिष्ठ स्थूणाराजो भवति । तस्मात्प्राङ् प्रक्रामति त्रीन्विकमां रतच्छङ्कुं निहन्ति सोऽन्तः पातः ॥१ ॥

तस्मान्मध्यमाच्छङ्कोः । दक्षिणा पञ्चदश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति सा दक्षिणा श्रोणिः ॥२ ॥

तस्मान्मध्यमाच्छङ्कोः । उदङ् पञ्चदश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति सोत्तरा श्रोणिः ॥३ ॥

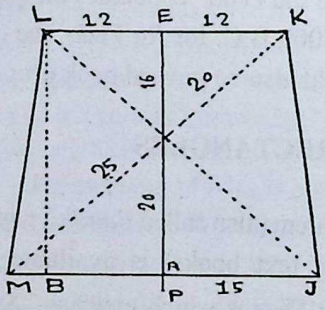
तस्मान्मध्यमाच्छङ्कोः । प्राङ् पट्त्रि ꣳशतं विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति स पूर्वार्धः ॥४ ॥

तस्मान्मध्यमाच्छङ्कोः । दक्षिणा द्वादश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति स दक्षिणोऽ ꣳसः ॥५ ॥

तस्मान्मध्यमाच्छंङ्कोः । उदङ् द्वादश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति स उत्तरोऽ ꣳस स एषा मात्रा वेदेः ॥६ ॥

अध्याय- ५ ब्राह्मण १

This means : From the tallest pole in the *s'ālā* move 3 *padas* (foot) towards east, and fix up a nail there. Call it "*antah pāta*". From the centre of this nail move 15 *padas* towards south, and fix up a nail there, which is called south (right) pelvic. From the centre of the nail move 15 *padas* towards north, and fix up a nail there, which is called north (left) pelvic. From the centre of the nail move 36 *padas* towards east, and fix up a nail there, which is called *Purvardha* (half-east point). From the centre of (this) nail move 12 *padas* towards north, and fix up a nail there which is called south (right) pectoral. From the centre of the nail move 12 *padas* towards north, and fix up a nail there, which is called north (left) pectoral. These are the *mātrās* (magnitudes) of the (*Mahā*) *vedi*.



Evidently, *Mahāvedī* is *JKLM* which is an isosceles trapezoid. *P* is the starting point and *A* = *antah pāta* where *PAE* is a line from west to east. Subsequent *mantras* give reasons for the choice of the lengths *MJ, LK* and *AE*. The vedi is compared with a woman whose lower part is wider than the upper part,

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since a woman is considered superior because of her womb which is responsible for creation (see [26], pages 429-431). Note that the *vedī* is loaded with Pythagorian triplets such as (12, 36, 39), (27, 36, 45).

Set II:

(ꣳ) स॒ व्वेद्यन्तात् । (त्सु) षट्त्रि ꣳ शत्प्रक्रमां प्राचीं व्वे॒दिं व्वि॒भिमीते त्रि ꣳ शतं पश्चात्तिर॒श्चीं च॒तुर्व्वि ꣳ शतिं पु॒रस्ता॒त्तन्नवति: सैषानव॒तिप्रक्रमा व्वे॒दिस्त॒स्याꣳ सप्त॒विधमग्निं व्वि॒दधाति ॥४ ॥

त॒दाहु: । कथ॒मेष॒ सप्त॒व्विध ऽएत॒या व्वे॒द्या स॒म्पद्यत ऽइ॒ति दु॒श वा ऽइ॒मे पु॒रुषे प्राणा॒श्चत्वार्स॒ङ्गान्या॒त्मा पञ्चदश ऽएव॑ द्वि॒तीय ऽएव॑ तृ॒तीये॒ पट्सु॒ पु॒रुषे॒षु नवतिर॒थैक: पु॒रुषो॒ ऽत्येति पा॒ङ्क्तो वै पु॒रुषो लो॒म त्व॒ङ्मा ꣳ स॒मस्थि म॒ज्जा पा॒ङ्क्तो ऽइ॒यं व्वे॒दिश्चत॒स्रो दि॒श ऽआत्मा॑ पञ्च॒म्येव॒मेष॒ सप्त॒विध ऽएत॒मा व्वे॒द्या स॒म्पद्यते ॥५ ॥

These *mantras* also describe the *Mahāvedī JKLM* (cf. [18], p. 37) but there is no mention of pole P (and hence of PA). However, there is one significant aspect concerning the measurements. In fact, it says that this is 90 *padas vedi*, i.e., a *vedi* of 90 units/feet. Note that $JM + AE + KL = 30 + 36 + 24 = 90$ and $JL + MK = 45 + 45 = 90$. Note also the triangle BLJ is a right triangle and $27^2 + 36^2 = 45^2$, i.e., $JB^2 + BL^2 = JL^2$. Seidenberg [21, p. 106] has rightly said confidently that : "I therefore regard it as certain that the *Śatapatha Brahmaṇa* knows the Theorem (of Pythagoras)". He however has not mentioned the 90-*pada* - *vedi* aspect.

SAVIŚESA

Śulbas, the manual for construction of altars have different approximation for $\sqrt{2}$. This suggests an understanding of difference between raional and irrational numbers, since *Vedic* priests knew that their (approximate) value of $\sqrt{2}$ was not exact (see [8, p. 43] and [20, p. 328]). In *Śulbas*, if one wishes to turn a circle into a square (see, for instance, Baudhāyana *Śulbas* I.60; *Āpastamba Śulbas* III.9; *Kātyāyana Śulbas* III.14), then one comes across a notable contribution of *Sulbas*, viz., a rational approximation of $\sqrt{2}$, which is generally termed as *Savis'esa*. The *Śulbas* give

(V) $2 = 1 + 1/3 + 1/3.4 - 1/3.4.34$

It may be mentioned that the commentator *Rāma* of *Naimisa* (near Lucknow)

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gave the following improved approximation (see Kapur [9, p. 85] and Khadilkar [11, p. 83]).

(R) $\sqrt{2} = 1+1/3 + 1\backslash 3.4 - 1/3.4.34 - 1/3.4.34.33 + 1/3.4.34.34$

Several proofs of (V) have been suggested by G. Thibout (cf. [14] and S.D. Khadilkar [11] (see also [9, p. 98]). Simple arithmetical and algebraic proofs may be given to (V) and (R). The latter may further be improved as follows :

(S) $\sqrt{2} = 1+1/3 + 1/3.4 - 1/3.4.3.4 - 1/3.4.34.33 + 1/3.4.34.34$
$+ 1/3.4.34.33.34.36$
(Herein 36 may also be replaced by 35)

The approximation (V), (R) and (S) are correct to 5, 7 and 8 places of decimals respectively.

The traditional Indian or *Vedic* mathematics from *Āryabhata* (476 A.D.) to *Mādhava* (1340 - 1425 A.D.) embodies many surprising and beautiful concepts and results in algebra, geometry, trigonometry, mensuration, theory of numbers, the basic concept of limit, calculus (differential and integral both) and Taylor's series for *sines* and *cosines* upto second powers. However, most studies of Indian Mathematics have unfortunately (in fact unfortunate to historians and scientific community) not been able to see its contributions to the scientific methods. Thanks to the West that its modern savants were the first to bring an appreciation of a portion of huge Indian mathematical and astronomical compositions and the origin of geometry into the notice of the scientific community. I trust there is still a lot to be investigated, and hope that the era of neglect is perhaps over.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 83-90

STUDIES ON POLLUTION REDUCTION POTENTIAL OF EICHHORNIACRASSIPES GROWN IN INDUSTRIAL EFFLUENT

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(Received 24-12-1991)

ABSTRACT

The present investigation was conducted to find out the pollution reduction efficiency of *Eichhornia crassipes* grown in industrial wasts of Indian drug pharmaceutical limited, Virbhadra. Results obtained during this investigation showed that plant growth was healthy, leaf numbers were increased four times within 30 days and was better than that of the plants grown in tap water (control IInd). There was considerable increase in D.O. level i.e. 1.5 (mg/lt) to 3.75 (mg.It) and BOD value was decreased i.e. 240.6 to 175.0 (mg/It). But at the same time this value was reversed where *Eichhornia* plants were grown in tap water i.e. control IInd. A declined trend was recorded in other parameters such as hardness, alkalinity and chloride etc.

Keywords : ***Eichhornia***, Pollution, D.O. BOD effluent.

INTRODUCTION

It is a well known fact that aquatic bodies are more susceptable for contamination due to various type of waste discharged into these. These waste includes industrial effluent, sewage and domestic water etc. Raw sewage contains various organic and inorganic materials such as cellulose, carbohydrates, fats protein urea and amines etc. Direct discharged of these polluted water often creat health problems in densly populated area of the country. Large number of waste are being discharged into aquatic bodies. These aquatic bodies contain smaller or larger number of weed dependig on nature of aquatic body. Generally aquatic weeds are still regarded by majority as "menace" and nuisance because they are not yet aware of the great potential and economic values of these profusely growing uncontrollable plants.

Eichhornia crassipes is one of the most abundantly fast growing

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weed in tropical and subtropical parts of the world. This fast growing troublesome weed is a good source of pulp, paper and human food. Mazid (6) reported some tribal in Bangladesh eat the flowers, young peduncle, and young petiols of this plant.

Aquatic weeds are good source of protein as such and these may be substitute for the diet harvivorus live stock. *Water-hyacinthis* also consumed by cattle but for good consuption it has to be mixed with right proportion of other feed i.e. 30%. Cattle feed consisting equal proportion of this weed with 5% molases supported good growth of cattle (Mazid (6). This contains protein from 9.14 to 14.37/dry weight. Mazid (5) reported that 672.00, 103.74 and 84.64 lb protein is produced monthly per acre by *E. crassipes, Salvinia* sps and *Azolla pinnata* respectively in Bangla desh.

Water hyacinth can also be used as manure. Compost prepared from aquatic weeds are more effective than cow dung compost for yield in several crops. Twenty two, 25.5 and 7.5 percent more yield was gained by using compost of *E. crassipes,* than cow dung manure in Rice. Onion and Turnip respectively Mazid (6). Besides these they are used in Sudan for bio-gas production. According to NASA. Water-hyacinth harvested from one hectare will produce more than 70,000 m3 of Bio-gas.

The major role of these aquatic weeds in aquatic body to reduce the pollutants. It has been reported by National Academy of Sciences U.S.A. that some aquatic weed can scavenage inorganic and some organic compounds from water. These weeds are capable of absorbing and incorporating the dissolved materials into their own structure, Reddy and Debusk (9) reported that eight aquatic macrophytes including *E. crassipes* can remove the N and P from nutrient enriched water.

Water hyacinth is a natural resource to reduce pollution load from waste water and can be utilized for treatment of waste water. Important contribution in these lines has been made by (Schultze, (1966), Scheffield, (11); Sinha and Sinha, (12); Miner, (7); Rao et al. (8). Dunigon et al. (4); Walverton, (14); Walverton and Donald, (13)

Like other aquatic bodies, Ganga also receives various kinds of pollutants throughout its course. Albiet water pollution board in India was framed in 1974 but due to several problems it could not be so effective.

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Keeping in view of the economy of the country and nutritional value as well as agricultural importance of the aquatic weed it was thought to make a preliminary study on pollution reduction potential of *E. crassipes* of larger leaves and their growth on industrial effluent under Indian environment.

MATERIALS AND METHODS

Plants of *Eichhornia crassipes* of larger leaves (Water hyacinth) used in the present investigation were collected from the local ponds. Industrial effluent was collected from Indian drug pharmaceutical limited (IDPL), a drug industry, located at Virbhadra Rishikesh, India. Fourty lts. effluent was filled in each cemented tanks. Pure eflluent (without dilution) was used for the experiment. Controls were of two types, control first was pure effluent without plants and control second was tap water with *Eichhornia* plants. Three replicate were used for each set.

Four plants of *Water hyacinth* were kept in each tank at begning of the experiment. Each plant contained seven leaves. Various physico chemical and biological characters of the effluent were evaluated *at* initial stage of the experiment. Analysis of the effluent was done as suggested by APHA [1]. A regular analysis of the effluent and observation on growth of the plants was made at interval of 15 days for the period of one month. The experiment was reconducted to confirm the findings.

RESULTS AND DISCUSSION

Results obtained during this investigation are given in the Table. In general there was no change in physico chemical characterestics of first control (Pure effluent without plants). However negligible reduction in alkalinity, hardness and BOD was recorded. Markedly reduction in various parameters was noted in pure effluent where water hyacinth

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plants were grown. Enhancement of dissolved oxygen from 1.5 to 3.75 (mg/lt) was found after 30 days interval. Profuse growth of the plants was found in pure effluent siamilar to the plants grown in tap water. Number of leaves were increased four times after one month. (Table).

It is very much obvious that *Eichhornia crassipes* of larger leaves is capable to reduce the polutants from water as it is evident with enhacement of dissolved oxygen in the effluent (Table). Albeit BOD was decreased gradually and dissolved oxygen was increased in the effluent but BOD was slightly increased in second control. Sixty percent reduction in BOD and suspended particles was recorded by Walverton etal. [14] when sewage was passed from wsater hyacinth plants. Further Walverton and Donald [13] reported that one acre of dens growth of waterhyacinth can treat waste water generated from 2000 people. Dinges [3], a scientist of NASA, demonstrated that this weed based filtration system can function effectively and such treatment plants for waste from municipal are under trial operation.

Increased in BOD in second control (Tap water with plant) is similar with findings of Macvea and Boyd [2]. They reported that water hyacinth decreased dissolved oxygen when it is grown in clean water. This is a peculiar feature of this plant that it reduces pollution load when grown in waste water but creates pollution when grown in clean water.

Four times increse of leaves after 30 days on effluent showes its rapid and profuse growth. This indicates that water hyacinth can grown well in effluent of such industries.

With the above findings and researches done by workers, on utility of this weed for waste treatment and in agriculture, it is desirable that this can be clultivated on stored waste water. After a definite time weed should be harvested regularly for compost preparation and cattle feed Mazid [5]. This will help in multiways at one time.

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Table-IV : Effect of Eichhornia Plants on I.D.P.L. effluent.

SI. No.	Experimental tanks	No of days	No of Plants	No of Leaves	pH	Chloride	Hard-ness	Alkali-nity	Conduc-tivity	D.O.	B.O.D.
1.	Effluents 100% with plants	Q	4.00	7.00	6.20	15.00	95.00	89.00	0.230	1.50	240.60
C 1 2.	Effluents 100%	0	-	-	6.20	15.00	95.00	89.00	0.230	1.50	240.60
C11 3.	Tapwter with plants	0	4.00	7.00	7.01	10.00	175.00	162.00	0.218	8.00	0.88
1.	Effluents 100% with plants	15	4.00	16.50	5.90	5.00	70.00	80.00	0.235	2.30	210.00
C1. 2.	Effluents 100%	15	-	-	6.00	15.00	95.00	84.00	0.228	1.00	238.00
C11.3.	Tapwater with plants	15	4.00	16.50	6.80	5.00	150.00	153.00	0.220	9.80	1.40
1.	Effluents 100% with plants	30	4.00	31.50	5.80	5.00	70.00	75.00	0.230	3.75	175.00
CI 2.	Effluents 100%	30	-	-	6.00	15.00	95.00	84.00	0.223	0.40	232.00
CII 3.	Tapwater with plants	30	4.00	30.60	6.70	-	150.00	148.00	-	10.40	2.50

Control Ist (Pure without effluent plants)
Control IInd (Tap water with plant)

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Developing countries like India should use such kinds of weeds (biological means) for treatment of waste water in the pattern of NASA. This would require proper weed management. A large number of aquatic weeds including waterhyacinth can be tested for treatment of municipal waste and compost preparation as well as bio gas production etc.

ACKNOWLEDGEMENT

Authors record their sincere thanks to the university authorities for their constant encouragement and facilities provided.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp- 91-94

A NOTE ON THE MODULAR SEMI-GROUP RING OF A FINITE IDEMPOTENT SEMIGROUP

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(Received 31.12.1991)

In [1] Johnson has proved when G is a group of prime power order p and K a field having only p-elements then G^*, the mod-p-envelope of G has a group structure. In this note we prove for an idempotent semi-group S of finite order having an identity 1 and K a field of characteristic p, S^* the mod p-envelope of S has a monoid structure, with non-trivial idempotents in it. For more properties about modular group rings please refer [1] and [2]. Just we define a semi-group ring of a semi-group S over a ring R analogous to group rings as follows:

Definition 1.
Let R be a ring (not necessarily commutative and not necessarily with a unity) and let S be a semi-group. We define the semi-group ring $R[S]$, to be the ring consisting of all finite formal sums $\Sigma\alpha_i x_i$ ($\alpha_i \in R$; $x_i \in S$) with the obvious definition of addition and with multiplication induced by the given multiplication in R and S according to the rule. $(\Sigma\alpha_i x_i)(\Sigma\beta_j y_j) = \Sigma(\alpha_i\beta_j) x_i y_j$ ($\alpha_i, \beta_j \in R$; $x_i, y_j \in S$). We take $(\Sigma\alpha_i x_i)$ to mean $\Sigma(\alpha_i)\ x_i$ ($\alpha_i \in R$; $x_i \in S$). If R has a unity 1_R then we impose the further condition $1_R x = x$ for all $x \in S$. In particular $R[S]$ can be viewed as a free left R-module. When R is a commutative field and S a monoid $R[S]$ is a semi- group algebra. Throughout this paper by S we mean a commutative idempotent semi-group under multiplication with multiplicative identity 1.

Example 1.
Let $K = (0, 1)$ and $S = \{1, g_1, g_2 \mid g_1^2 = g_1, g_2^2 = g_2, g_1g_2 = g_2g_1\}$ be a semigroup. KS the semigroup ring of S over K is given by $KS = \{0, 1, g_1, g_2, g_1g_2, 1+g_1, 1+g_2, 1+g_1g_2, g_1 + g_2, g_1 + g_1g_2, g_2 + g_1g_2, 1+ g_1 + g_2, 1+ g_1 +g_1g_2, 1+ g_2+ g_1g_2, g_1+ g_2 +g_1g_2.\ 1 + g_1+ g_2 +g_1g_2\}$. $S^* = 1+U$ *where* $U = \{\Sigma\alpha_i g_i \mid \Sigma\alpha_i = 0\}$ ($\alpha_i \in K$, $g_i \in S$); $S^* = 1+U = 1+ \{g_1 + g_2,\ 1+g_1,\ 1+g_2,\ 1+g_1g_2, g_1 + g_1g_2, g_2 + g_1g_2,\ 1 + g_1+ g_2 +g_1g_2,\ 0\} = \{g_1, g_2, g_1g_2, 1 + g_1+ g_2,\ 1, 1 + g_1 +g_1g_2,\ 1 +g_2+g_1g_2,\ g_1+ g_2 +g_1g_2\}$ clearly S^* *is* an idempotent semi-group with identity and order of S^* is 8.

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Definition 2.
Let S be a senti-group with identity and K a field of characteristic p. KS the semi-group ring of S over K, then S^*, the mod p-envelope of S is $1+U$ where $U = \{ \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0,\ \alpha_i \in K;\ S_i \in S \}$.

Proposition 3.
Le $K = (0,1)$ be the field of integers modulo 2 and $S = \{1, g_i | g_i^2 = g_i\ g_i g_j = g_j g_i\}$ be an idempotent semigroup of order n. Then S^* is a monoid in which every element is an idempotent and order of S^* is 2^{n-1}.

Proof.
$S^* = 1+U$ where $U = \{ \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0 \}$. Here $U = \{0$, elements taken two at a time, elements taken four at a time,..., elements taken n at a time if n is even or elements taken $n-1$ at a time if n is odd). Thus order of S^* is 2^{n-1}. This is easy to verify $1 \in S^*$ and every element of S^* is an idempotent as characteristic of K is two.

Example 2.
Let $K = (0, 1,2)$ be the field of integers modulo 3 and $S = \{1,\ g_1 g_2 \mid g_1^2 = g_1,\ g_2^2 = g_2,\ g_1 g_2 = g_2 g_1\}$ Then S^* is not an idempotent semigroup but contains identity and idempotents. For $S^* = 1+U = 1+ \{ g_1 + g_2 + g_1 g_2,\ g_1 + 1 + g_1 g_2,\ g_1 + g_2 + 1,\ 1 + g_2 + g_1 g_2,\ 2 + 2 g_1 + 2 g_1 g_2,\ 2 g_1 + 2 g_2 + 2 g_1 g_2,\ 2 + 2 g_2 + 2 g_1 g_2,\ 2 + 2 g_1 + 2 g_2,\ g_1 + g_2 + 2 + 2 g_1 g_2,\ 2 g_1 + 2 g_2,\ + g_1 g_2 + 1,\ 2 g_1 + 2 + g_2 + g_1 g_2,\ 2 g_1 + 2 g_1 g_2 + g_2 + 1,\ 2 g_1 g_2 + 2 g_2 + g_1 + 1,\ 2 g_2 + 2 + g_1 g_2 + g_1,\ 2 g_1 + g_2,\ 2 g_1 + 1,\ 2 g_1 + g_1 g_2,\ g_1 + 2 g_2,\ g_1 + 2 g_1 g_2,\ g_1 + 2,\ g_2 + 2,\ 2 + g_1 g_2,\ 2 g_1 g_2 + 1,\ 2 g_1 g_2 + g_2,\ g_1 g_2 + 2 g_2,\ 2 g_1 + 1,\ 0\}$ Now S^* contains idempotents and not every element of S^* is an idempotent. $1 \in S^*$ with order of $S^* = 27$.

Proposition 4.
Let $Z_p = (0, 1,\ldots,p-1)$ be the field of integers mod-p, p a prime and S be a commutative idempotent semi group of order n with identity 1. Then the following conditions are true in S^*.
(1) S^*, the mod p envelope of S has P^{n-1} elements
(2) $S \underline{C} S^*$
(3) S^* has no zero divisors.

Proof.
Since $S^* = 1+U$ where $U = \{ 0, \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0,\ \alpha_i \in K$ and $s_i \in S\}$ we have order

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of S^* to be p^{n-1}. Clearly SCS^*

If $\alpha \in S^*$ be such that there exists a $\beta \in S^*$ with $\alpha\beta = 0$ then we will arrive at a contradiction. Let $\alpha = \Sigma \alpha_i s_i$ and $\beta = \Sigma\beta_j s_j$ with $\Sigma\alpha_i = \Sigma\beta_j = 1$ $\alpha\beta = \Sigma\alpha_i s_i \ \Sigma\beta_j s_j = \Sigma(\alpha_i\beta_i)\, s_i s_j$ if $\alpha\beta = 0$ we must have $\Sigma\alpha_j\beta_j = 0$. Obviously SCS^* as every element of S is contained in S^* (Since $(p-1) + s_i \in U$ for all $s_i \in S$ so $1 + (p-1) + s_i \in S^*$ ie $s_i \in S^*$), S^* has no zero divisors. For if S^* contains zero divisors say α, β such that $\alpha.\beta = 0$; then we must have $\Sigma\alpha_i$ $\Sigma\beta_j = \Sigma\alpha_i\beta_j = 0$ (where $\alpha = \Sigma\alpha_i s_i$ and $\beta = \Sigma\beta_i s_i$) which is impossible as $\Sigma\alpha_i = 1$ and $\Sigma\beta_j = 1$, we have $\Sigma\alpha_i\beta_j = 1$ as both α and β are in S^*. Thus S^* has no zero divisors.

Example 4.
Let $K = (0,1)$ and $S = < s,\ 1 \mid s^3 = s >$. S^* is a semi group having an idempotent and identity. For $KS = \{ 0,\ 1,\ s,\ s^2,\ 1+s,\ 1+s^2,\ 1+s+s^2,\ s+s^2\}$, Now $S^* = 1 + U = 1 + \{0,\ 1 + s,\ 1+s^2,\ s+s^2\} = \{1, s, s^2, 1+s+s^2\}$ Now $s^2 \in S^*$ is an idempotent of S^*.

Theorem 5.
Let $K = (0, 1)$ and S be a semigroup with 1 and an element s such that $s^p = s$ where p is any integer. Then S^* is a semigroup with 1and order of S^* is 2^{p-1}.

Proof obvious

Example 5.
Let $K = (0,1,2)$ and $S = \{1, s \mid s^3 = s\}$. The $KS = \{0, 1, 2, s, s^2, 2s, 2s^2, 1+s, 1+s^2, 1+s+s^2, s+s^2, 2+2s, 2+2s^2, 2+2s+2s^2, 2s+2s^2, 1+2s+2s^2, 2+ s+2s^2, 2+s^2+2s, 2+s+s^2, 1+s^2+2s, 1+2s^2+s, s+2s^2, 1+2s^2, 1+2s, 2+s^2, 2+s^2, 2+s, 2s+s^2\}$. $S^* = 1+U = 1+\{1+s+s^2, 2+2s+2s^2, 2+s, 0, 2s+s^2, 2+s^2, 1+2s^2, 1+2s, s+2s^2\}$. $S^* = \{1, 2+s+s^2, 2s+2s^2, s, s^2, 2+2s^2, 1+2s+s^2\}$ S^* is a semigroup with one has elements of order two and order of S^* is 9. Thus

Theorem 6. Let $K = (0, 1,..., p-1)$; p a prime and $S = \{s, 1 \mid s^p = s\}$. Then S^* is a semigroup of order p^{p-1} with identity 1.

Proof Obvious.

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We are not able to say anything when S is a general semigroup.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp- 95-104

STRESS DISTRIBUTION IN A NON - HOMOGENEOUS CYLINDRICALLY ANISOTROPIC ELASTIC CYLINDER

P. K. Chaudhuri* & Subrata Datta*

(Received 03.01.1992)

ABSTRACT

This paper deals with the determination of the radial displacement and relevant stresses in a cylinder on the basis of three dimensional linear theory of elasticity. The material of the cylinder is orthotropic with cylindrical anisotropy and, in addition, is continuously non-homogeneous, with mechanical properties varying along the radius. The variations of stresses along the radius have been shown graphically.

Mathematical Subject Classification (1991) : 76

Keywords: Orthotropic material, Cylindrical anisotropy, Elastic parameter, Nonhomogeneous material, Hypergeometric function.

INTRODUCTION

The solution for radial deformation and corresponding stresses in a cylindrical shell of homogeneous isotropic elastic media under the action of internal and external pressure was obtained by Lame' as quoted in Love [5].

Since last few decades many investigators have been paying attentions to problems in elasticity in which the material is no longer homogeneous. Due to complications in the governing differential equations these problems are sometimes solved by numerical methods. Grief and Chou [3] have considered such a problem. They have solved the vibration problem of an anisotropic non-homogeneous cylindrical shell without considering any longitudinal extension. Roy [7] has found the radial deformation and stresses in a cylindrically anisotropic non-homogeneous material under the influence of normal pressure on both the boundaries, allowing uniform longitudinal extension. In our present investigation we have considered the problem of stress distribution in an elastic

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cylinder of non-homogeneous material. The medium is assumed to be such that the elastic coefficients at any point vary exponentially with an arbitrary power of the radial distance of the point considered. Due to this assumption the governing differential equation reduces to a confluent hypergeometric equation by suitable transformation. The expression for the radial displacement and stresses in the homogeneous case have been shown to be the limiting values of the corresponding expressions of our results.

FORMULATION OF THE PROBLEM

Taking the axis of anisotropy as the z-axis and using cylindrical coordinate (r, θ, z) the stress-strain relations for cylindrically orthotropic material take the form (cf. Lekhnitskii [4], Armenakas and Ritz [2])

$$\sigma_r = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}$$

$$\sigma_\theta = c_{21}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz}$$

$$\sigma_z = c_{31}e_{rr} + c_{32}e_{\theta\theta} + c_{33}e_{zz}$$

$$\tau_{\theta z} = c_{44}e_{\theta z}, \quad \tau_{rz} = c_{55}e_{rz}, \quad \tau_{r\theta} = c_{66}e_{r\theta} \tag{1}$$

where $c_{ij} = c_{ji}$, $(i,j = 1,2,3,)$

The strain components in terms of the displacement (u,o,w) are

$$e_{rr} = \partial u/\partial r, \; e_{\theta\theta} = u/r, \; e_{zz} = \partial w/\partial z$$

$$e_{rz} = \partial u/\partial z + \partial w/\partial r, \; e_{r\theta} = e_{\theta z} = 0 \tag{2}$$

Let the nonhomogeneity of the material be characterised by the variations of the elastic parameters c_{ij} according to the following law

$$c_{ij} = \lambda_{ij} e^{-kr^m}, \; k > 0 \tag{3}$$

Where λ_{ij}'s are the prescribed parameters of the material concerned and $m(\neq 0)$ is a real number. The case $m = 0$ corresponds to the homogeneous medium.

For the case

$$w = ez \tag{4}$$

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where e is a constant and $\partial u/\partial z = 0$, we have a state of plane-strain with a uniform longitudinal extension superposed.

In view of (4) $\tau_{rz} = 0$ and the stress equation of equilibrium in the absence of the body forces reduce to

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \tag{5}$$

Using equations (2), (3) and (4) in (1) we get the stresses in terms of the displacement, then substitution of the stresses in (5) yields the equation of equilibrium as

$$r^2 \frac{d^2u}{dr^2} + (1 - mkr^m) r \frac{du}{dr} - \left(\frac{\lambda_{22} + \lambda_{12} kmr^m}{\lambda_{11}}\right) u = \frac{\lambda_{13}}{\lambda_{11}} emkr^{m+1} - \left(\frac{\lambda_{13} - \lambda_{23}}{\lambda_{11}}\right) er \tag{6}$$

If we assume that the cylinder $a \le r \le b$ is under the action of internal and external pressures, the boundary conditions may be put as

$$\sigma_r = -p_o \quad \text{at} \quad r = a$$
$$\sigma_r = -p_1 \quad \text{at} \quad r = b \tag{7}$$

Our problem is to solve equation (6) satisfying (7).

Solution of the problem

In view of the transformation

$$z = kr^m, \; u = z^n V, \; n = s/m \tag{8}$$

the equation (6) reduces to

$$z \frac{d^2V}{dz^2} + (\gamma - z)\frac{dV}{dz} - \alpha V = z^q(\xi z + \eta) \tag{9}$$

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where

$$\alpha = (s + \lambda_{12}/\lambda_{11})p$$

$$\gamma = 2n+1$$

$$q = p - n - 1$$

$$s = \sqrt{(\lambda_{22}/\lambda_{11})} = mn$$

$$p = 1/m$$

$$\xi = \lambda_{13}/\lambda_{11}\ epk^{-p}$$

$$\eta = (\lambda_{23} - \lambda_{13})/\lambda_{11}\ ep^2 k^{-p}$$

The complementary equation

$$z\frac{d^2V}{dz^2} + (\gamma - z)\frac{dV}{dz} - \alpha V = 0 \qquad (10)$$

of the differential equation (9) is a confluent hypergeometric equation whose solution is known (Abramowitz and Stegun [1]) for integral as well as for non integral values of γ, being given by

$$V = AM(\alpha\ ;\ \gamma\ ;\ z) + B\ \phi(z) \qquad (11)$$

In (11) $M(\alpha\ ;\ \gamma\ ;\ z)$ is the confluent hypergeometric function such that

$$M(\alpha;\ \gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{z^n}{n!} \qquad (12)$$

Where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$ with $(\alpha)_o = 1$.

The series on the right hand side of (12) is known to be convergent (Rainville[6]) for all finite z unless $\gamma = 0$ or a negative integer.

The function $\varphi(z)$ in (11) depends on γ as follows :

$\varphi(z) = z^{1-\gamma}\ M(\alpha+1-\gamma\ ;\ 2-\gamma\ ,z)$ when γ is not an integer

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$$= \frac{(-1)^{K+1}}{K!\,\Gamma(\alpha-K)}\,[M(\alpha;\,K+1;\,z)\,\ln z$$

$$+ \sum_{r=0}^{\infty} \frac{(\alpha)_r\, z^r}{(k+1)_r\, r!}\,\{\psi(\alpha+r) - \psi(1+r) + \psi(1+K+r)\}]$$

$$+ \frac{(K-1)!}{\Gamma(\alpha)}\, z^{-K}\, M(\alpha-K;\,1-K;\,z)_K \qquad (13)$$

When γ is an integer and $= K+1$, $(K = 0,1,2,\ldots)$, the last function being the sum to K terms. It is to be interpreted as zero when $K = 0$, and $\psi(\alpha) = \dfrac{\Gamma'(\alpha)}{\Gamma(\alpha)}$.

The particular solution of (9) is given by

$$V = Z^{\beta}\,[a_0 + a_1 z\ {}_2F_2\,(1, \alpha+\beta+1;\,\beta+2,\,\beta+\gamma+1\,;z)]$$

$$= f(z),\ \text{say}$$

Where

$$a_0 = \frac{\eta}{\beta(\beta+\gamma-1)}$$

$$a_1 = \frac{1}{(\beta+1)(\beta+\gamma)}\left[\xi + \frac{\eta(\alpha+\beta)}{\beta(\beta+\gamma+1)}\right]$$

$\beta = p-n$ and ${}_2F_2\,(a, b;c,d;z)$ is the generalised hypergeometric function such that the series represented by ${}_2F_2\,(a,b;c,d;z)$ is convergent for all finite z, provided, of course, that neither c nor d is zero or a negative integer.

Thus the general solution of (9) is known. Hence using (8) the displacement u is given by

$$u = z^n\,[AM\,(\alpha\,;\,\gamma\,;\,z) + B\,\phi\,(z) + f\,(z)] \qquad (14)$$

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Substituting (14) in (2) we get the components of strain. The components of stress may be obtained from (1) together with (3) and (8) as

$$\sigma_r = A\, h_1(z) + B\, h_2(z) + g(z) \tag{15}$$

$$\sigma_\theta = A\, h_3(z) + B\, h_4(z) + f_1(z) \tag{16}$$

$$\sigma_z = A\, h_5(z) + B\, h_6(z) + f_2(z) \tag{17}$$

where

$$f_1(z) = [z^{n-p}\{c_4 f(z) + c_6 z f'(z)\} + e\,\lambda_{23}] \exp(-z),$$

$$f_2(z) = [z^{n-p}\{c_7 f(z) + c_9 z f'(z)\} + e\,\lambda_{33}] \exp(-z),$$

$$g(z) = [z^{n-p}\{c_1 f(z) + c_3 z f'(z)\} + e\,\lambda_{13}] \exp(-z),$$

$$h_1(z) = z^{n-p}[c_1 M(\gamma - \alpha; \gamma; -z) + c_2 z M(\gamma - \alpha; \gamma + 1; -z)],$$

$$h_2(z) = z^{n-p}[c_1 \phi(z) + c_3 z \phi'(z)] \exp(-z),$$

$$h_3(z) = z^{n-p}[c_4 M(\gamma-\alpha; \gamma; -z) + c_5 z M(\gamma - \alpha; \gamma + 1; -z)],$$

$$h_4(z) = z^{n-p}[c_4 \phi(z) + c_6 z \phi'(z)] \exp(-z),$$

$$h_5(z) = z^{n-p}[c_7 M(\gamma-\alpha; \gamma; -z) + c_8 z M(\gamma - \alpha; \gamma + 1; -z)],$$

$$h_6(z) = z^{n-p}[c_7 \phi(z) + c_9 z \phi'(z)] \exp(-z),$$

$$c_1 = k^p (mn\lambda_{11} + \lambda_{12}),$$

$$c_2 = \frac{\alpha}{\gamma} m\lambda_{11} k^p,$$

$$c_3 = m\lambda_{11} k^p,$$

$$c_4 = k^p (mn\lambda_{12} + \lambda_{22}),$$

$$c_5 = \frac{\alpha}{\gamma} m\lambda_{12} k^p,$$

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$$c_6 = m\lambda_{12}\, k^{p},$$

$$c_7 = k^{p}\,(mn\lambda_{13} + \lambda_{23}),$$

$$c_8 = \frac{\alpha}{\gamma}\; m\,\lambda_{13}\; k^{p},$$

$$c_9 = m\lambda_{13}\, k^{p}.$$

Finally the boundary conditions (7) determine A and B as

$$A = [g_2(z_2)\, h_2(z_1) - g_1(z_1) h_2(z_2)]/\Delta$$

$$B = [h_1(z_2)\, g_1(z_1) - h_1(z_1) g_2(z_2)]/\Delta \qquad (18)$$

where

$$\Delta = h_2(z_1)\, h_1(z_2) - h_1(z_1)\, h_2(z_2)$$

$$g_1(z) = -p_0 - g(z),\; g_2(z) = -p_1 - g(z)$$

$$z_1 = ka^{m},\; z_2 = kb^{m}$$

Substitution of (18) in (14), (15), (16) and (17) determines displacement and stresses completely.

A partial check of the results presented in this discussion has been made by making $m \to 0$ and $k \to 0+$. Thus we see from (3) that λ_{ij} are the elastic coefficients in the homogeneous case. Here we shall confine our attention only to the case in which γ is not an integer. Choosing appropriate $\phi(z)$ from (13) and letting m $\to$ 0 and k $\to$ 0 + we get the displacement and stresses as

$$u = \frac{ab}{(s\lambda_{11} + \lambda_{12})}\left[\frac{q_0}{b}\left(\frac{r}{b}\right)^{s} - \frac{q_1}{a}\left(\frac{r}{a}\right)^{s}\right] \Big/ \left[\left(\frac{b}{a}\right)^{s} - \left(\frac{a}{b}\right)^{s}\right]$$

$$+ \frac{ab}{(\lambda_{12} - s\lambda_{11})}\left[\frac{q_1}{a}\left(\frac{a}{r}\right)^{s} - \frac{q_0}{b}\left(\frac{b}{r}\right)^{s}\right] \Big/ \left[\left(\frac{b}{a}\right)^{s} - \left(\frac{a}{b}\right)^{s}\right]$$

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$$+ \frac{(\lambda_{23} - \lambda_{13})er}{\lambda_{11} - \lambda_{12}}. \tag{19}$$

$$\sigma_r = \frac{\frac{ab}{r}\left\{\left[\frac{q_0}{b}\left(\frac{r}{b}\right)^s - \frac{q_1}{a}\left(\frac{r}{a}\right)^s\right] + \left[\frac{q_1}{a}\left(\frac{a}{r}\right)^s - \frac{q_0}{b}\left(\frac{b}{r}\right)^s\right]\right\}}{\left[\left(\frac{b}{a}\right)^s - \left(\frac{a}{b}\right)^s\right]}$$

$$+ \frac{(\lambda_{23} - \lambda_{13})}{\lambda_{11} - \lambda_{22}}(\lambda_{11} + \lambda_{12})e + \lambda_{13}e \tag{20}$$

$$\sigma_\theta = \frac{\frac{ab}{r}\left\{\begin{array}{l}\left(\frac{s\lambda_{12} + \lambda_{22}}{s\lambda_{11} + \lambda_{12}}\right)\left[\frac{q_0}{b}\left(\frac{r}{b}\right)^s - \frac{q_1}{a}\left(\frac{r}{a}\right)^s\right] \\ + \left(\frac{\lambda_{22} - s\lambda_{12}}{s\lambda_{12} - \lambda_{11}}\right)\left[\frac{q_1}{a}\left(\frac{a}{r}\right)^s - \frac{q_0}{b}\left(\frac{b}{r}\right)^s\right]\end{array}\right\}}{\left[\left(\frac{b}{a}\right)^s - \left(\frac{a}{b}\right)^s\right]}$$

$$+ \left(\frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}}\right)(\lambda_{12} + \lambda_{22})e + \lambda_{23}e \tag{21}$$

$$\sigma_z = \frac{\frac{ab}{r}\left\{\begin{array}{l}\left(\frac{s\lambda_{13} + \lambda_{23}}{s\lambda_{11} + \lambda_{12}}\right)\left[\frac{q_0}{b}\left(\frac{r}{b}\right)^s - \frac{q_1}{a}\left(\frac{r}{a}\right)^s\right] \\ + \left(\frac{\lambda_{23} - s\lambda_{13}}{\lambda_{12} - s\lambda_{11}}\right)\left[\frac{q_1}{a}\left(\frac{a}{r}\right)^s - \frac{q_0}{b}\left(\frac{b}{r}\right)^s\right]\end{array}\right\}}{\left[\left(\frac{b}{a}\right)^s - \left(\frac{a}{b}\right)^s\right]}$$

$$+ \left(\frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}}\right)(\lambda_{13} + \lambda_{23})e + \lambda_{33}e \tag{22}$$

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$$q_0 = p_0 + \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}} (\lambda_{11} + \lambda_{12})e + \lambda_{13}e$$

$$q_1 = p_1 + \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}} (\lambda_{11} + \lambda_{12})e + \lambda_{13}e$$

These results are in complete agreement with the results of the homogeneous case.

If the material of the structure be isotropic and homogeneous the material constants λ_{ij} become

$$\lambda_{11} = \lambda_{22} = \lambda_{33} = \lambda + 2\mu\,,\ \lambda_{12} = \lambda_{13} = \lambda_{23} = \lambda$$

In this case $s = 1$ and the displacement and the stress components have the same expressions as those given in Love [5].

Numerical Results

Numerical computations of stresses have been made by taking

$\lambda_{11} = 40.75$ $\lambda_{22} = 50.61$ $\lambda_{33} = 42.63$ $\lambda_{44} = 15.66$
$\lambda_{55} = 19.14$ $\lambda_{66} = 19.00$ $\lambda_{23} = 12.76$ $\lambda_{13} = 12.18$
$\lambda_{12} = 18.27,$

which are the material constants for topaz. Here λ_{ij}'s are expressed in term

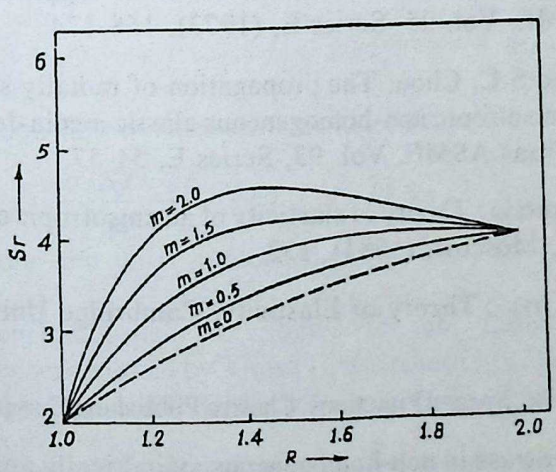

Fig.1 : Variation of Sr with R

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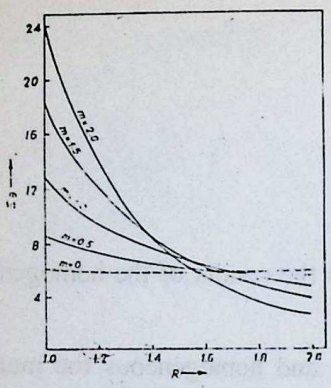

Fig.2 : Variation of S_θ with R

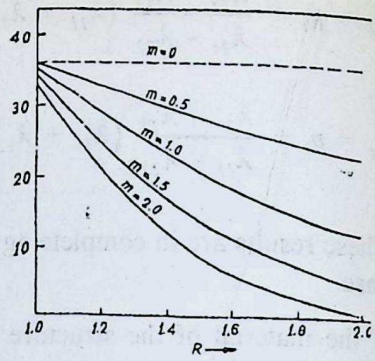

Fig.3 : Variation of S_z with R

of a unit stress of $10^6\ lb/(in)^2$. In addition, we assume that $\delta = 2 \times 10^6\ lb(in)^2$ and write $p_0 = \delta e$, $p_1 = 2\delta e$. With this assumption, we plot the values of $S_r = -\sigma r/e$, $S_\theta = -\sigma\theta/e$ and $S_z = -\sigma z/e$, against $R = (r/a)$ in $1 \leq R \leq 2$, in Fig. 1, Fig. 2 and Fig.3 respectively. In our calculation we have considered $m = 0.0, 0.5, 1.0, 1.5, 2.0$. From the figures it is observed that the stresses change regularly with variations of m.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 105-118

A STUDY OF SHAPE OF ACTIVATED COMPLEX IN REACTION BETWEEN POTASSIUM PEROXODISULPHATE AND POTASSIUM IODIDE

Fahim Uddin* & Huma Kazmi*

(Record 17.07.92)

ABSTRACT

The rate constants of the reaction between potassium peroxodisulphate and potassium iodide have been measured in a series of water acetone mixtures at 30± 0.1°C and d40 ± 0.1°C at various ionic strengths (μ). The radii of activated complexes for single sphere and double sphere models have been calculated from the slopes of linear plots of logarithm of rate constant at zero ionic strength (k_0) versus reciprocal of dielectric constant ($1/\varepsilon$). Results show that the description of the activated complex could best be given by the single sphere model.

An attempt has also been made to evaluate the values of electrostatic contributions to the free energies changes ($\Delta G^{\neq}_{c.s}$).

INTRODUCTION

The reaction between peroxodisulphate and iodide ions takes place in two steps [1,2]

$$S_2O_8^{--} + I^- \longrightarrow IS_2O_8^{--} \qquad \text{slow ... (1)}$$

$$IS_2O_8^{---} + I^- \longrightarrow I_2 + 2SO_4^{--} \qquad \text{fast ... (2)}$$

This reaction follows second order kinetics:

$$\frac{-d[S_2O^{--}{}_8]}{dt} = k[S_2O^{--}{}_8]\ [I^-]z$$

where 't' is time and 'k' is rate constant.

The influence of dielectric constant on the rate of ionic reactions having similar charges was studied by Amis [3], Scatchard [4], Laidler [5], Laidler-Eyring [6], Christiansen [7], Ghaziuddin - et al. [8-10] and Fahim Uddin et

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al. [11-13]. Ghaziuddin - et al. [14] and Fahim Uddin et al. [15] also studied the reaction between peroxodisulphate-iodide ions in ethanol water and methanol water mixtures respectively and found that the activated complex is single sphere for both the media i.e. ethanol water and methanol water mixtures.

In the present work, the reaction between potassium peroxodisulphate and potassium iodide has been studied to confirm whether the activated complex consists of single sphere or double sphere model.

EXPERIMENTAL

Potassium peroxodisulphate, potassium iodide, sodium thiosulphate, potassium dichromate, starch, acetone used were of E. Merck. Double distilled water was used throughout the course of experiments.

Dielectric constants of the medium were varied by changing proportions of water acetone mixtures. The composition of water acetone mixtures *(W/W)* were 8. 08%, 12.25%, 16.51%, 20.07% and 25.32% and respective values of dielectric constants were taken from Akerlof [16-17].

The rate constants of the reaction were measured by a titrimetric method. Calculated volumes of $K_2S_2O_8$ and KI were taken in 25 ml volumetric flasks and diluted up to the mark. These two diluted solutions were kept in a thermostatic bath (Type K-33 Haake, Karlsruhe, Germany) at constant temperature for half an hour. Then these two solutions were mixed and the time was recorded. A 5 ml portion of this reaction mixture was withdrawn after each interval of 5 minutes. The mixture was chilled to freeze the reaction and titrated against a standard solution of sodium thiosulphate. 2 drops of starch solution was added as an indicator.

RESULTS AND DISCUSSIONS

The rate constants were calculated using the integrated form of the rate expression for a second order reaction with the same initial concentrations i.e.,

$$k = \frac{1}{t} \cdot \frac{x}{a(a-x)} \qquad \text{.....(4)}$$

where *'a'* is the initial concentration of potassium peroxodisulphate and potassium iodide, x is the concentration of the product at time *'t'* and *(a-x)* is the remaining concentration of the reactant at time *'t'*.

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Plots of *[x/a(a-x)]* against time 't' were drawn at different ionic strengths and composition of solute i.e. acetone water mixtures. A representative plot of *[x/a(a-x)]* versus *'t'* at ionic strength (μ) = 7.84 x 10^{-2} mil.dm^{-3}in 10% acetone water mixtures at 40°C is shown in figure - 1. From the slopes of the plots, rate constants were evaluated. The values of rate constants for the

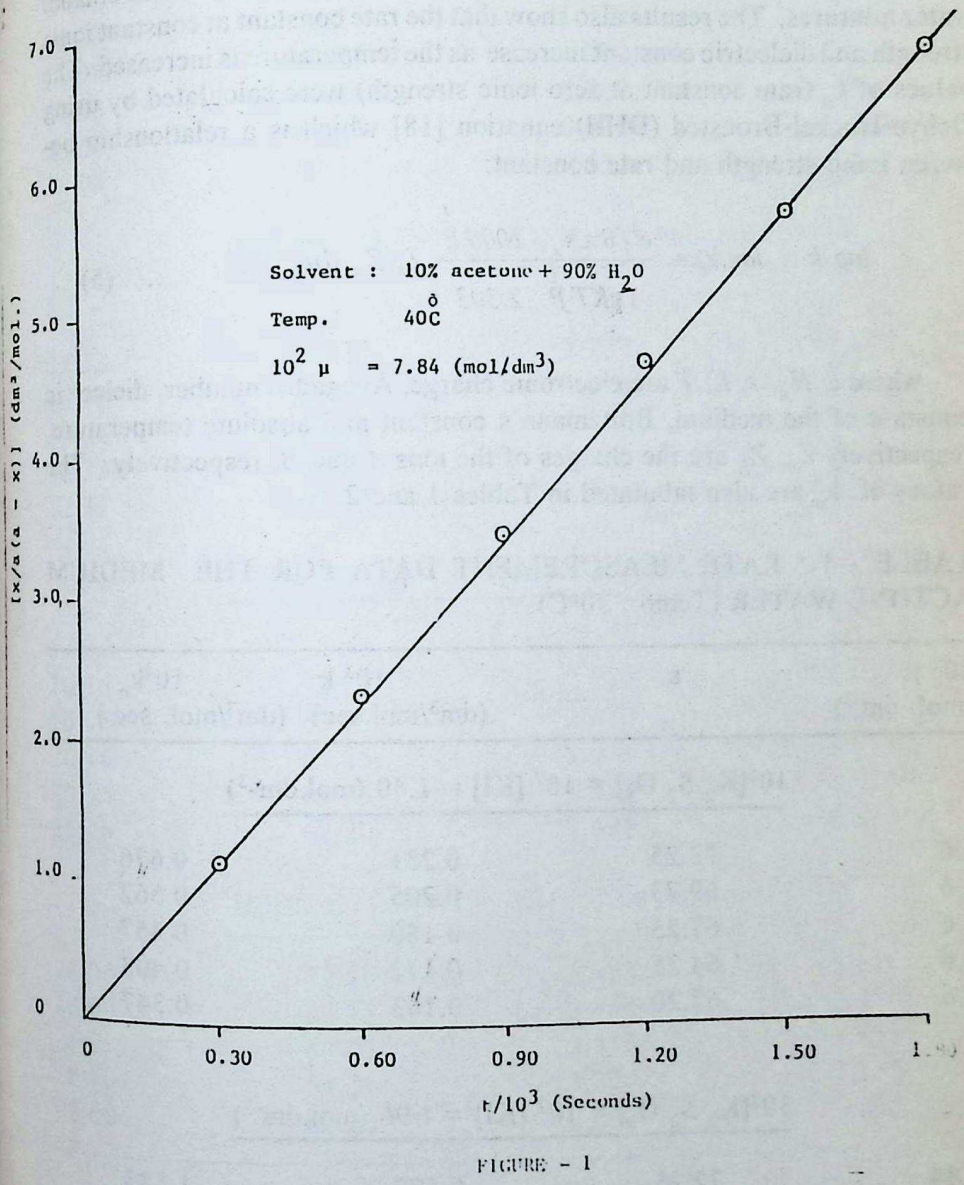

FIGURE - 1

Fig 1 : Plot of *[x/a (a-x)]* versus *'t'* at 40^0C

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reaction between peroxo-disulphate and iodide ions in acetone water mixtures at 30° and 40°C are summarized in Tables 1 and 2 respectively.

It was also noted that for the same value of dielectric constant, the rate constant increased with increasing ionic strength. These results are in agreement with those reported earlier [14] for the same reaction in ethanol water mixtures. The results also show that the rate constant at constant ionic strength and dielectric constant increase as the temperature is increased. The values of k_o (rate constant at zero ionic strength) were calculated by using Debye-Huckel-Bronsted (DHB) equation [18] which is a relationship between ionic strength and rate constant:

$$\log k = \log k_o + \frac{e^3 (8\pi N_A / 1000)^{\frac{1}{2}}}{(\varepsilon KT)^{\frac{3}{2}} .2.303} . Z_A . Z_B . \sqrt{\mu} \qquad .....(5)$$

where e, N_A, ε, K, T are electronic charge, Avogadro number, dielectric constant of the medium, Boltzmann's constant and absolute temperature, respectively Z_A, Z_B are the charges of the ions A and B, respectively. The values of k_o are also tabulated in Tables 1 and 2.

TABLE - 1 : RATE MEASUREMENT DATA FOR THE MEDIUM ACTONE-WATER (Temp : 30°C)

$10^2 \mu$ (mol. dm^{-3})	ε	10^3 k (dm^3/mol.sec)	$10^4 k_o$ (dm^3/mol. sec.)
	$10^2[K_2 S_2 O_8] = 10^2$ [KI] = 1.40 (mol.dm^{-3})		
5.6	72.25	0.231	0.676
5.6	69.75	0.205	0.562
5.6	67.25	0.180	0.457
5.6	64.25	0.172	0.403
5.6	62.20	0.163	0.347
	$10^2[K_2 S_2 O_8] = 10^2$ [KI] = 1.96 (mol.dm^{-3})		
7.84	72.25	0.677	1.585
7.84	69.75	0.625	1.349
7.84	67.25	0.419	0.832

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7.84	64.25	0.342	0.616
7.84	62.20	0.278	0.447

$10^2[K_2 S_2 O_8] = 10^2 [KI] = 2.52$ (mol.dm^{-3})

10.08	72.25	1.820	3.467
10.08	69.75	1.090	1.905
10.08	67.25	0.963	1.549
10.08	64.25	0.872	1.259
10.08	62.20	0.653	0.830

$10^2[K_2 S_2 O_8] = 10^2 [KI] = 3.08$ (mol.dm^{-3})

12.32	72.25	2.167	2.089
12.32	69.75	1.318	1.950
12.32	67.25	1.042	1.380
12.32	64.25	0.983	1.148
12.32	62.20	0.876	0.849

$10^2[K_2 S_2 O_8] = 10^2 [KI] = 5.04$ (mol.dm^{-3})

20.16	72.25	2.777	2.692
20.16	69.75	2.670	2.291
20.16	67.25	2.031	1.513
20.16	64.25	1.538	0.977
20.16	62.20	1.495	0.813

$10^2[K_2 S_2 O_8] = 10^2 [KI] = 7.00$ (mol.dm^{-3})

28.00	72.25	5.483	3.516
28.00	69.75	3.958	2.188
28.00	67.25	3.513	1.659
28.00	64.25	3.230	1.267
28.00	62.20	2.602	0.832

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TABLE - 2 RATE MEASUREMENT DATA FOR THE MEDIUM ACETONE WATER (Temp : 400C)

$10^2 \mu$ (mol. dm-3) sec.)	ε	10^3 k (dm^3/mol.sec)	$10^4 k_o$ (dm^3/mol.
	$10^2[K_2 S_2 O_8] = 10^2 [KI] = 1.40$ (mol.dm-^{3})		
5.60	69.00	1.967	5.623
5.60	66.75	1.823	4.898
5.60	64.25	1.592	3.981
5.60	62.00	1.102	2.521
5.60	59.75	1.061	2.239
	$10^2[K_2 S_2 O_8] = 10^2 [KI] = 1.96$ (mol.dm^3)		
7.84	69.00	3.816	8.709
7.84	66.75	3.683	7.762
7.84	64.25	2.516	4.786
7.84	62.00	1.616	2.818
7.84	59.75	1.482	2.344
	$10^2[K_2 S_2 O_8] = 10^2 [KI] = 2.52$ (mol./dm^3)		
10.08	69.00	4.762	8.851
10.08	66.75	4.333	7.431
10.08	64.25	3.333	5.129
10.08	62.00	3.000	4.169
10.08	59.75	3.563	3.162
	$10^2[K_2 S_2 O_8] = 10^2 [KI] = 3.08$ (mol./dm^3)		
12.32	69.00	6.066	9.332
12.32	66.75	5.867	8.318
12.32	64.25	3.533	4.467
12.32	62.00	3.492	3.935
12.32	59.75	3.432	3.388

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$10^2[K_2 S_2 O_8] = 10^2 [KI] = 5.04$ (mol./dm^3)

20.16	69.00	8.108	7.499
20.16	66.75	6.900	5.623
20.16	64.25	5.283	3.715
20.16	62.00	4.398	2.692
20.16	59.75	4.385	2.399

$10^2[K_2 S_2 O_8] = 10^2 [KI] = 7.00$ (mol./dm^3)

28.00	69.00	8.903	5.370
28.00	66.75	7.433	3.890
28.00	64.25	6.583	2.884
28.00	62.00	5.000	1.738
28.00	59.75	4.842	1.493

The effect of dielectric constant on rate constant can be explained by the Laidler-Eyring equation [6] for single and double sphere models respectively. The equations are expressed as followed.

$$\ln k_0 = \ln k_\infty - \frac{e^2}{2\varepsilon KT}\left[\frac{Z^2_A}{r^{\neq}} - \frac{Z^2_A}{r_A} - \frac{Z^2_B}{r_B}\right] \quad \text{(for single sphere mode) ...(6)}$$

$$\ln k_0 = \ln k_\infty - \frac{Z_A . Z_B . e^2}{KT r_{AB}} \cdot \frac{1}{\varepsilon} \quad \text{(for double sphere model)...(7)}$$

where k_∞ is the rate constant at zero ionic strength and infinite dielectric constant. $r^{\neq}$ and r_{AB} are the radii of the activated complex for the single model and the double sphere model, respectively, and r_A, r_B are radii of the ions A and B, respectively. The plots of logarithm of rate constant at zero ionic strength ($log\ k_o$) against the reciprocal of the dielectric constant of the medium ($1/\varepsilon$) at various ionic strengths and temperatures were straight lines with negative slopes. A representative graph is shown in Figure - 2.

The experimental values of $r^{\neq}$ and r_{AB} calculated from the slopes of straight lines using equations (6) and (7) are tabulated in Table - 3.

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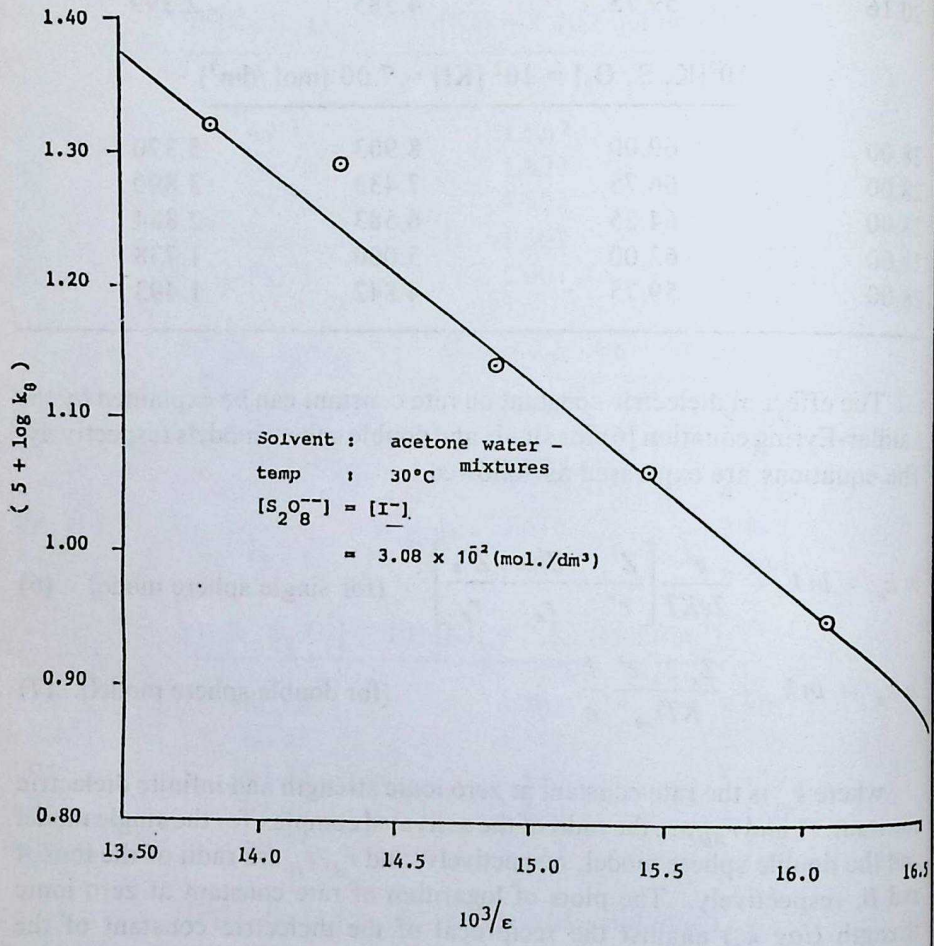

FIGURE - 2

Fig. 2 : Plot of log k_o versus '$1/\varepsilon$' in aqueous acetone medium at 30°C.

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TABLE - 3 : CALCULATED VALUES OF RADII OF ACTIVATED COMPLEX

$10^2[K_2S_2O_8]$ (mol/dm^3)	10^2[KI] (mol/dm^3)	$r^{\neq}$ (Å)	r_{AB} (Å)
		Temp : 30°C	
1.40	1.40	3.04	3.74
1.96	1.96	2.23	1.86
2.52	2.52	2.25	1.89
3.08	3.08	2.70	2.77
5.05	5.04	2.26	1.91
7.00	7.00	2.19	1.81
		Average	Average
		2.44 ± 0.37	2.33 ± 0.83
		Temp : 40°C	
1.40	1.40	2.57	2.48
1.96	1.96	2.13	1.72
2.52	2.52	2.49	2.33
3.08	3.08	2.44	2.22
5.05	5.04	2.30	1.98
7.00	7.00	2.20	1.83
		Average	Average
		2.35 ± 0.2	2.09 ± 0.32

The radius of the peroxodisulphate ion was estimated on the assumption that the volume of the molecule of a compound which is supposed to be spherical is a sum of volumes of individual ions consituting the compound as follows :

$$V_{S_2O_8^{2-}} = 2V_{S6} + + 8V_{O^{2-}}$$

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where V is the volume of the ion.

$$\frac{4}{3}\pi r^3_{S_2O_8^{2-}} = 2\left(\frac{4}{3}\pi r^3_S 6 + + 8\left(\frac{4}{3}\pi r^3_{O^{2-}}\right)\right.$$

$$r_{S_2O_8^{2-}} = 2.80\ \text{Å}$$

$r_{I^-} = 2.16\ \text{Å}$ (taken from Emeleus [19] .....(11)

The radius of activated complex for single sphere model ($r^{\neq}$) is calculated on the basis of assumption that the volume of single sphere activated complex ($V^{\neq}$) is the sum of volume of reactants.

$$V^{\neq} = V_{I^-} + V_{S_2O_8^{2-}} \quad .....(12)$$

$$r^{\neq} = 3.176\ \text{Å} \quad .....(13)$$

The radius of activated complex for double sphere model is calculated assuming that the radius of activated complex is the sum of the radii of reactants.

$$r_{AB} = r_{S_2O_8^{2-}} + r_{I^-} \quad .....(14)$$

$$= 2.80 + 2.16 = 4.96\ \text{Å} \quad .....(15)$$

In all calculations the values of radii of individual ions were taken from Emeleus [19] and confirmed by Wilson [20]. It has been concluded from the comparison of average experimental values of $r^{\#}$ and r_{AB} with their theoretical values that the shape of the activated complex could best be given by single sphere model for reaction between peroxodisulphate and iodide ions because the value of $r^{\#}_{expt}$ is more closer to the theoretical value of single sphere model. The results show that change in temperature has no effect on the shape of

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activated complex. The values of $r^{\#}$ and r_{AB} are slightly less than the values given by Ghaziuddin *et al*. [14]. This may be due to the volatile nature of acetone.

Electrostatic contributions to the changes in free energy of activation $\Delta G^{\#}_{es}$ in the formation of activated complex for single sphere model were evaluated using the following expression [5] :

$$\Delta G^{\neq}_{e.s} = \frac{N_A e^2}{2}\left[\frac{(Z_A + Z_B)^2}{r^{\neq}} - \frac{Z^2 A}{r_A} - \frac{Z^2 B}{r_B}\right] \qquad \text{.....(16)}$$

The values of $\Delta G^{\neq}_{es}$ at different temperatures i.e. 30^0 and $40\,^0$C and ionic strengths as a function of dielectric constant are summarised in Table - 4.

ABLE - 4 : ELECTROSTATIC CONTRIBUTION TO $\Delta G^{\neq}_{es}$

$I0^2\,\mu$ (mol/dm^3)	Dielectric constant (ε)				
	72.25	69.75	67.25	64.75	62.20
			Temp : 30°C		
5.60	1.028	1.065	1.105	1.147	1.194
7.84	2.062	2.136	2.215	2.300	2.394
10.08	2.027	2.100	2.178	2.262	2.235
12.32	1.386	1.436	1.489	1.547	1.610
20.16	2.010	2.083	2.160	2.243	2.335
28.00	2.133	2.209	2.291	2.380	2.477
			Dielectric constant (ε)		
	69.00	66.75	64.25	62.00	59.75

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Temp : 40°C				$\varepsilon_{e.s}$(KJ/mole)	
5.60	1.621	1.676	1.741	1.804	1.872
7.84	2.349	2.429	2.523	2.615	2.713
10.08	1.735	1.793	1.863	1.930	2.003
12.32	1.809	1.870	1.943	2.014	2.089
20.16	2.035	2.103	2.186	2.265	2.350
28.00	2.214	2.289	2.378	2.464	2.555

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp-119-122

ON A LAPLACE - HARDY TRANSFORMATION

J. M. C. Joshi* & H. S. Nayal*

(Received 04.09.1992)

ABSTRACT

In this paper we shell give a few funuamental theorems that depict certain properties of Laplace-Hardy transformation that seem to be interesting and significant.

INTRODUCTION

In the present paper for the Classical Laplace-Hardy transformation. we shall use the notation:

(1). $\phi\,[f: p.q] = pq\int_0^{\infty}\int_0^{\infty} e^{-\mu x q y} C_{\nu}(pxqy)\, f(px, qy)\, dxdy$

where $f(px, qy)$ is suitably chosen function under a certain set of condition on ν and α given by Pathak and Pandey, [5].
and

(2). $C_{\nu}(z) = \cos(a\pi) J_{\nu}(z) + \sin(a\pi)\, Y_{\nu}(z)$

$J_{\nu}(z)$ and $Y_{\nu}(Z)$ being Bessel functions [(2). p.38] of the first and second kind respectively. and (2) valid under certain conditions given by Erdelyi [2].

Throughout this work we use the notation:

(3). $\phi\,[f: p,q] \underset{\sim}{C_{\nu}} f(px, qy)$.

The C_{ν}-function is defined by (2). and if we take $a = 0$. we obtain $C_{\nu}(z) = J_{\nu}(z)$ And If $a = 1/2$. we obtain $C_{\nu}(z) = y_{\nu}(z)$. And if $\nu = 1/2$. we obtain $C_{\nu}(z = [(1/2)\pi z]^{-1/2} \sin(z - a\pi)$. So by using the conditions on 'ν' and 'a' we obtain. some special cases of Hardy's transform.

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Theorem

If $\phi_i [f_i : p,q] \underset{\bullet\bullet}{\overset{C_\nu}{=}} f_i(px, qy),\ i = 1,2.$

then

$$(4). \quad \int_0^\infty \int_0^\infty \phi_1[f_1 : p,q] f_2(px, qy) \frac{dp}{p}\frac{dp}{q}$$

$$= \int_0^\infty \int_0^\infty f_1(px, qy)\ \phi_2[f_2 : p,q] \frac{dp}{p}\frac{dp}{q}$$

provided that the integral involved are absolutely convergent.

Proof

By putting the value of $\phi_1 [f_1 : p,q]$ on the left hand side of (4), we have

$$\int_0^\infty \int_0^\infty \{ pq \int_0^\infty \int_b^\infty e^{-\mu px - \nu qy} C_\nu(pxqy) f_1(px, qy)\, dx dy \} f_2(px, qy) \frac{dp}{p}\frac{dp}{q}$$

Now changing the order of integration, we get

$$= \int_0^\infty \int_0^\infty \{ pq \int_0^\infty \int_0^\infty e^{-\mu px - \nu qy} C_\nu(px, qy) f_2(px, qy) \frac{dp}{p}\frac{dp}{q} \} f_1(px, qy)\, dx dy.$$

$$= \int_0^\infty \int_0^\infty f_1(px, qy) \{ \int_0^\infty \int_0^\infty e^{-\mu px - \nu qy} C_\nu(pxqy) f_2(px, qy)\, dp\, dq \}\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty f_1(px, qy) \{ xy \int_0^\infty \int_0^\infty e^{-\mu px - \nu qy} C_\nu(pxqy) f_2(px, qy)\, dp\, dq \} \frac{dx}{x}\frac{dy}{y}$$

$$= \int_0^\infty \int_0^\infty f_1(px, qy)\ \phi_2[f_2 : p,q] \frac{dx}{x}\frac{dy}{y}$$

$$= \int_0^\infty \int_0^\infty f_1(px, qy)\ \phi_2[f_2 : p,q] \frac{dp}{p}\frac{dq}{q}$$

Hence the theorem.

Theorem 2

Let $\phi [f : p,q] \underset{\bullet\bullet}{\overset{C_\nu}{=}} f(px, qy) = g(p,q)$

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and $\phi[h:p,q] \underset{\bullet\bullet}{\overset{C_\upsilon}{=}} h(px,qy) = \psi(p,q)$

then

(5). $$\int_0^\infty \int_0^\infty \psi(u,v) f(ux,vy) (uv)^{-1} du\, dv$$

$$= \int_0^\infty \int_0^\infty g(u,v) h(ux,vy) (uv)^{-1} du\, dv$$

provided that either of the integral in (5) is absolutely convergent and the ϕ-transformation of $|f(ux, vy)|$ and $|h(ux,vy)|$ exist.

Proof

In view of theorem 1:

(6). $$\psi(u,v) = uv \int_0^\infty \int_0^\infty e^{-uxvy} C_\upsilon(uxvy)\, h(ux,vy)\, dx\, dy$$

Now putting the value $\psi(u,v)$ on the left hand side of (5) :

$$\int_0^\infty \int_0^\infty \{uv \int_0^\infty \int_0^\infty e^{-uxvy} C_\upsilon(uxvy)\, h(ux,vy)\, dx\, dy\} f(ux,vy)(uv)^{-1} du\, dv$$

changing the order of integration:

$$= \int_0^\infty \int_0^\infty \{uv \int_0^\infty \int_0^\infty e^{-uxvy} C_\upsilon(uxvy)\, f(ux,vy)\, (uv)^{-1} du\, dv\} h(ux,vy) dx\, dy$$

$$= \int_0^\infty \int_0^\infty \{xy \int_0^\infty \int_0^\infty e^{-uxvy} C_\upsilon(uxvy)\, f(ux,vy)\, du\, dv\} h(ux,vy)\, (xy)^{-1} dx\, dy$$

$$= \int_0^\infty \int_0^\infty g(x,y)\, h(ux,vy)\, (xy)^{-1}\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty g(u,v)\, h(ux,vy)\, (uv)^{-1}\, du\, dv.$$

Hence the theorem.

Presently numerical inversion of certain integral transforms are being studied in order to make them computer oriented. [6].

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Journal of Natural & Physical Sciences Vol. 5(1991-94) pp- 123-137

ON A NONLINEAR FUNCTIONAL INTEGRODIFFERENTIAL SYSTEM

M. B. Dhakne*

(Received 08.09.1992)

ABSTRACT

The aim of the present paper is to study the existence, uniqueness and other properties of the solutions of a nonlinear functional integrodifferential system of more general type by using monotone iterative method, comparison principle, the notion of upper and lower solutions and integral inequality established by Pachapatte.

Mathematical Subject Classification: 47HO5 47H10

Keywords and Phrases: Integrodifferential System, Monotone iterative technique.

INTRODUCTION

In what follows, we denote by R^n the *n-dimensional Euclidean space* with the Euclidean norm $\|\cdot\|$. Let $I_o = [-r, 0]$, $I = [t_0, t_o+T]$, $J = [t_0 - r, t_0+T]$, $r > 0$, $t_o \geq 0$ and $T > o$ ar econstants. Let $C = C[I_0, R^n]$ be the Banach space of continuous functions defined on I_o with supremum norm $\|\cdot\|_c$. We use the notation $< v, w >$ to denote the segment in $C[J, R^n]$ defined by $<v, w> = \{x \in C[J, R^n] : v(t) \leq x(t) \leq w(t), t \in J\}$. If x is a continous function from J to R^n and $t \in I$, then x_t denotes the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta \in I_o$. In the present paper, we investigate the existence, uniqueness and other properties of the solution of following nonlinear functional intergro differential system

$$x'(t) = f\left(t, x(t), x_t, \int_{t_0}^{t} k(t, s, x(s), x_s)ds\right), \quad t \in I. \tag{1.1}$$

$$x_{t_0}(\theta) = \phi(\theta), \quad \theta \in I_0$$

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where $x : J \to R^n$, $K \in C [I \times I \times R^n \times C, R^n]$, $f \in C [I \times R^n \times C \times R^n, R^n]$ and $\theta \in C [I_0, R^n]$ is a given function.

The problems of existence, uniqueness and other properties of solutions of special forms of the system (1.1) have been studied by many authors in the literature by using different techniques, for example, see [2,4,6,7,9,10,12] and the references given therein. In [9], V. Lakshmikantham and B.G. Zhang have proved the existence of extremal solution of the equation (1.1) when k = 0 by using monotone iterative technique. B.G. Pachpatte [12] has established the existence of extermal solutions of the system (1.1) without functional argument. The main tools employed in our analysis are based on the monotone iterative method, comparision principle, the notion of upper and lower solutions and integral inequality established by Pachpatte.

We organize the paper as follows: In section 2, we present the preliminaries and statement of our main results, the proofs of which are contained in sections 3 and 4.

PRELIMINARIES AND STATEMENT OF RESULTS

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses on the functions involved in (1.1) that will be used in our subsequent discussion. Throughout this paper, without further mention, we assume that all inequalities between vectors are componentwise

Definition 1. The function $w \in C [J,R^n] \cap C' [I, R^n]$ is called an *upper solution* of (1.1.) if

$$w'(t) \geq f\left(t, w(t), w_t, \int_{t_0}^{t} k(t,s,w(s), w_s)ds \right), \quad t \varepsilon I$$

$$w_{t_0}(\theta) \geq \phi(\theta), \quad \theta \in I_0$$

Similarly, a function $v \in C [J,R^n] \cap C' [I,R^n]$ is called a *lower solution of* (1.1) if

$$v'(t) \leq f\left(t, v(t), v_t, \int_{t_0}^{t} k(t,s,v(s), v_s)ds \right), \quad t \in I,$$

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$v_{t_0}(p) \leq \phi(\theta). \quad \theta \in I_0$

Definition 2. The functions α and β, $\alpha, \beta \in C[J,R^n] \cap C'[I,R^n]$ are called minimal and maximal solutions of (1.1) respectively, if every other solution $x \in C[J,R^n] \cap C'[I,R^n]$ of (1.1) satisfies the relation

$$\alpha(t) \leq x(t) \leq \beta(t), \; t \in J.$$

Definition 3. The solution $x(t)$ of (1.1) is said to be uniformly slowly growing if for every $\gamma > 0$ there exists a constant Q, possibly depending on γ, such that the indequality

$$\|x_t\|_C \leq Q \|\phi\|_C . e^{\gamma(t-t_0)}, \; t \geq t_0$$

holds for $\|\phi\|_k < \infty$

we require the following lemmas in our further discussion.

Lemma 1. (See, [9,p.228]). Assume that $u \in C[J,R] \cap C''[I,R]$ and

$$u'(t) \leq -Mu(t) - N \int_{-r}^{0} u_t(s)ds,$$

$t \in I$. Further assume that

i) either $u(t_0) \leq u_{t_0}(s) \leq 0$, $s \in I_0$ and $(M + Nr)\, T \leq 1$
or

ii) $u_{t_0}(s) \leq 0$, $s \in I_0$, $u \in C'([t_0 - r, t_0], R)$ and

$$u'(t) \leq \frac{\lambda}{T + r}, \; t \in [t_0 - r, t_0]$$

where $\min_{[t_0 - r, t_0]} u(t) = -\lambda$, $\lambda \geq 0$ and $(M + Nr)(T + r) \leq 1$. Then $u(t) \leq 0$ on I.

Lemma 2. (See [11. p.758)]. *Let u(t), m(t) and n(t) be real* valued

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nonnegative continuous functions defined on R^+, for which the inequality

$$u(t) \leq u_0 + \int_0^t m(s)\, u(s)\, ds + \int_0^t m(s) \left[\int_0^s n(\tau) u(\tau)\, d\tau \right] ds.$$

holds for all $t \in R^+$, where u_0 is a nonnegative constant. Then

$$u(t) \leq u_0 \left[1 + \int_0^t m(s) exp \left\{ \int_0^s \left(m(\tau) + n(\tau) \right) d\tau \right\} ds \right].$$

for all $t \in R^+$

For convenience, we list the following hypotheses.

(H_1) The functions $v, w \in C. [J, R^n] \cap C^1 [I, R^n]$ with $v(t) \leq w(t)$ on J are lower and upper solutions of (1.1).

(H_2) For each $i, 1 \leq i \leq n$, $k_i (t, s, x, \phi)$ is *monotone nondecreasing* in x and ϕ for fixed t and s, and $f_i(t, x, \phi, u)$ is *monotone nondecreasing* in u for fixed t, x and ϕ.

(H_3) For each $i, 1 \leq i \leq n$, there exist constants $M, N, \geq 0$ such that

$$f_i (t, x, \phi, u) - f_i (t, \bar{x}, \bar{\phi} u)$$

$$\geq - M(x_i - \bar{x}_i) - N \int_0^0 \left[\phi_i(s) - \bar{\phi}_i(s) \right] ds$$

whenever $v(t) \leq \bar{x} \leq x \leq w(t)$ and $v_1 \leq \bar{\phi} \leq \Phi \leq w_1$ for $t \in I$.

(H_4) The difference $v_{i_0} - \phi$, $\phi - w_{i_0}$ satisfy either the assumption (i) or (ii) of lemma 1

(H_5) There exist nonegative constants M_1, M_2, M_3, M_4 and M_5 such that

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(2.1) $\| K(t,s,x,\phi) - k(t,s,\bar{x}\,\bar{\phi}) \| \leq M_1 \| x - \bar{x} \| + M_2 \| \phi - \bar{\phi} \|_c$

(2.2) $\| f(t,x,\phi,u) - f(t,\bar{x}\,\bar{\phi},\bar{u}) \|$
$\leq M_3 \| x - \bar{x} \| + M_4 \| \phi - \bar{\phi} \|_C + M_4 \| u - \bar{u} \|$

(H_6) For $t, s \in [t_0, \infty)$, $\phi \in C$, $x, u \in R^n$,

(2.3) $\| K(t,s,x,\phi) \| \leq L_i(s) [\| x \| + \| \phi \|_C]$

(2.4) $\| f(t,x,\phi,u) \| \leq L_2(t) [\| x \| + \| \phi \|_C + \| u \|]$

where $L_1, L_2 \in C([t_0, \infty), [t_0, \infty))$
and

(2.5) $\int_{t_0}^{\infty} L_1(s)ds < \infty, \int_{t_0}^{\infty} L_2(s)ds < \infty.$

(H_7) For $t, s \in [t_0, \infty]$, $\gamma > 0$, $\phi \in C$, $x, u \in R^n$,

(2.6) $\| K(t,s,x,\phi) \| \leq e^{-\gamma(s-t)} L_1(s) [\| x \| + \| \phi \|_C]$

(2.7) $\| f(t,x,\phi,u,) \| \leq e^{-\gamma t} L_2(t) [\| x \| + \| \phi \|_C + \| u \|]$

where $L_1, L_2 \in C([t_0, \infty), [t_0, \infty))$ and (2.5) holds.

Our main results are established in the following theorems.

THEOREM 1. Let the hypotheses (H_1) - (H_4) hold. Then there exists monotone nondecreasing sequence $\{v_n\}$ and monotone non-increasing sequence $\{w_n\}$ such that $v_n \to \alpha$, $w_n \to \beta$ as $n \to \infty$ uniformly on I and that α, β are minimal and maximal solutions of the system (1.1) respectively. That is, if x is any solution of (1.1) in $\langle v, w \rangle$, then

$$v \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq \alpha \leq x \leq \beta \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w$$

THEOREM 2. Let the hypotheses (H_1) - (H_5) hold. Then the maximal solution $\beta(t)$ and the minimal solution $\alpha(t)$ obtained in Theorem 1 coincide on J, that is $\beta(t) = \alpha(t)$, for $t \in J$.

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In the following theorems, we assume that solutions of the system (1.1) exist on $[t_o - r, \infty)$.

THEOREM 3. Let the hypothesis (H_6) holds. Then any solution $x(t)$ of the system (1.1) is bounded on $[t_o - r, \infty)$.

THEOREM 4 Let the hypothesis (H_7) holds. Then any solution $x(t)$ of the system (1.1) is uniformly slowly growing.

Remark 1. We note that in [8], G.S. Ladde and B.G. Pachpatte, have established existence of the solution of the equation of type (1.1) by using different conditions and techniques. It is also important to note that M.B. Dhakne and B.G. Pachpatte in [5] have investigated the problems of boundedness, stability, asympotic behaviour and other properties of the solutions of an abstract functional integro-differential equation by using the semigroups method.

PROOFS OF THEOREMS 1 AND 2

For any $\eta \in C[J, R^n]$ such that $v \leq \eta \leq w$ on J, we consider the following linear system

$$x'(t) = f\left(t, \eta(t), \eta_t, \int_{t_0}^{t} k(t, s, \eta(s), \eta_s) ds\right) - M[x(t) - \eta(t)]$$

$$- N \int_{-r}^{0} [x_t(s) - \eta_t(s)] ds, \quad t \in I, \tag{3.1}$$

$$x_{t_0}(\theta) = \phi(\theta), \theta \in I_0$$

Clearly, the linear system (3.1) possesses a unique solution $x(t)$ on J, for each η. We, now, define a mapping A by $A\eta = x$.

We prepare a lemma which forms the basis of our main result.

Lemma 3. Let the hypotheses (H_1) - (H_4) hold. Then

(i) $v \leq Av$, $w \geq Aw$ on J;

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(ii) A is a monotone operator on the segment $<v,w>$

Proof : Set $Aw = x$ where x is the unique solution of (3.1) corresponding to w, and $z_i(t) = x_i(t) - w_i(t)$ for each i, $1 < i < n$. Then, we have

$$z'_i(t) = x'_i(t) - w'_i(t)$$

$$\leq f_i\left(t, w(t), w_t, \int_{t_0}^{t} k(t,s,w(s), w_s)ds\right) - M[x_i(t) - w_i(t)]$$

$$-N\int_{-r}^{0}\left[x_{i_t}(s) - w_{i_t}(s)\right]ds - f_i\left(t, w(t), w_t, \int_{t_0}^{t} k(t,s,w(s), w_s)ds\right)$$

$$= -Mz_i(t) - N\int_{-r}^{0} z_{i_t}(s)\,ds$$

and $z_{i_{to}} = \phi_i - w_{i_{to}} \leq 0$. By an application of lemma 1, $z_i(t) = x_i(t) - w_i(t) \leq 0$ on I, that is, for each i, $1 \leq i \leq n$, $w_i(t) \geq x_i(t)$ for $t \in J$. Hence, $w(t) \geq x(t) = Aw(t)$ on J. Similarly, we show that $v \leq Av$ on J. This proves(i).

Let $p, q \in < v, w >$ and $p \leq q$. Suppose that $Ap = x$ and $Aq = y$. For each i, $1 \leq i \leq n$, set $z_i(t) = x_i(t) - y_i(t)$. Then

$$z_i'(t) = x_i'(t) - y'_i(t)$$

$$= f_i(t, p(t), p_t, \int_{t_0}^{t} k(t, s, p(s), p_s)\,ds) - M[x_i(t) - p_i(t)]$$

$$-N\int_{-r}^{0}[x_{i_t}(s) - p_{i_t}(s)]\,ds - f_i(t, q(t), q_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\,ds)$$

$$+ M[y_i(t) - q_i(t)] + N\int_{-r}^{0}\left[y_{i_t}(s) - q_{i_t}(s)\right]ds$$

$$\leq f_i(t, p(t), p_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\,ds$$

$$- f_i(t, q(t), q_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\,ds) - M[x_i(t) - p_i(t)]$$

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$$- N \int_{-r}^{0} [\, x_{i_t}(s) - p_{i_t}(s)] \, ds + M [y_i(t) - q_i(t)]$$

$$+ N \int_{-r}^{0} [\, y_{i_t}(s) - q_{i_t}(s)] \, ds$$

$$= - Mz_i(t) - N \int_{-r}^{0} z_{i_t}(s) \, ds$$

and $z_{i_0} = \phi_i - \phi_i = 0$. By lemma 1, $z_i(t) = x_i(t) - y_i(t) \leq 0$ on I, that is for each i, $1 \leq i \leq n$, $x_i(t) \leq y_i(t)$ for $t \in J$. Hence, $x(t) \leq y(t)$ on J. Thus, A is a monotone operator on $\langle v, w \rangle$. This completes the proof of lemma 3.

In order to prove Theorem 1, define the sequences $\{ v_n \}$ and $\{ w_n \}$ as follows :

$$v_1 = Av, \; v_n = Av_{n-1}, \; n = 2,3,\ldots$$

$$w_1 = Aw, \; w_n = Aw_{n-1}, \; n = 2,3,\ldots$$

on J. by an application of lemma 3, we get

$$v \leq v_1 \leq v_2 \leq \ldots \leq v_n \leq w_n \ldots \leq w_2 \leq w_1 \leq w$$

on J. Using standard arguments (see [1], [3]), it is easy to show that

$$\lim_{n \to \infty} v_n = \alpha \quad \text{and} \quad \lim_{n \to \infty} w_n = \beta$$

uniformly on J and that α, β are solutions of the system (1.1) on J.

Let $x(t)$ be any solution of the system (1.1) such that $v(t) \leq x(t) \leq w(t)$ on J. For each i, $1 \leq i \leq n$, set $z_i(t) = v_{1_i}(t) - x_i(t)$. Then

$$z'_i(t) = v_{1_i}'(t) - x'_i(t)$$

$$= f_i(t, v(t), v_{t'} \int_{t_0}^{t} k(t, s, v(s), v_s) \, ds) - M [v_{1_i}(t) - v_i(t)]$$

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$$-N \int_{-r}^{0} [v_{l_{i_t}}(s) - v_{i_t}(s)]\, ds - f_i(t, x(t), x_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\, ds)$$

$$\leq f_i(t, v(t), v_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\, ds),$$

$$- f_i(t, x(t), x_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\, ds)$$

$$- M[v_{l_i}(t) - v_i(t)] - N \int_{-r}^{0} [v_{l_{i_t}}(s) - v_{i_t}(s)]\, ds$$

$$\leq M[x_i(t) - v_i(t)] + N \int_{-r}^{0} [x_{i_t}(s) - v_{i_t}(s)]\, ds$$

$$- M[v_{l_i}(t) - v_i(t)] - N \int_{-r}^{0} [v_{l_{i_t}}(s) - v_{l_i}(s)]\, ds$$

$$= - Mz_i(t) - N \int_{-r}^{0} z_{i_t}(s)\, ds$$

and $z_{i_{t_0}} = v_{l_{t_0}} - x_{i_{t_0}} = \phi_i - \phi_i = 0$ Then by an application of Lemma 1, we obtain $z_i(t)\ v_{l_i}\ 0$ on J.That is : for each i, $1 \leq i \leq n$, $v_{l_i}(t) \leq x_i(t)$, $t \in J$. Hence $v_1(t) \leq x(t)$ for $t \in J$. By induction, it can be shown that $v_n \leq x$ for every $n \geq 2, 3, \ldots$ .Similarly, $x \leq w_n, n = 1, 2, \ldots$ .Therefore, $v_n \leq x \leq w_n$ and hence $\alpha \leq x(t) \leq \beta$ on J.That is : α, β are minimal and maximum solutions of the system (1.1) respectively. This completes the proof of the Theorem 1.

The function $\beta(t)$ and $\alpha(t)$ are solutions of the system (1.1). so we have,

$$\beta(t) = \phi(0) + \int_{t_0}^{t} f(s, \beta(s), \beta_s, \int_{t_0}^{\tau} k(s, \tau, \beta(\tau), \beta_\tau)\, d\tau) ds,\ t \in I,$$

(3.2)

$$\beta(t) = \phi(t - t_0),\ t_0 - r \leq t \leq t_0$$

and

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$$\alpha(t) = \phi(0) + \int_{t_0}^{t} f(s, \alpha(s), \alpha_s, \int_{t_0}^{s} k(s, \tau, \alpha(\tau), \alpha_\tau)\, d\tau) ds, \quad t \in I,$$

(3.3)

$$\alpha(t) = \phi(t - t_0), \quad t_0 - r \le t \le t_0$$

Define $z(t) = \| \beta(t) - \alpha(t) \|$, $t \in J$, Clearly, $z(t) = 0$ for $t_0 - r \le t \le t_0$. Now for $t \in I$, we obtain from (3.2), (3.3) and using (2.1) and (2.2),

$$z(t) = \| \beta(t) - \alpha(t) \|$$

$$\le \int_{t_0}^{t} \| f(s, \beta(s), \beta_s, \int_{t_0}^{s} k(s, \tau, \beta(\tau) \beta_\tau)\, d\tau)$$

$$- f(s, \alpha(s), \alpha_s, \int_{t_0}^{s} k(s, \tau, \alpha(\tau) \alpha_\tau) d\tau \| \, ds$$

$$\le M_3 \int_{t_0}^{t} \| \beta(s) - \alpha(s) \|_C \, ds + M_4 \int_{t_0}^{t} \| \beta_s - \alpha_s \|_C \, ds$$

$$+ M_1 M_5 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta(\tau) - \alpha(\tau) \|_C \, d\tau \, ds + M_2 M_5 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

$$\le M_3 \int_{t_0}^{t} \| \beta_s - \alpha_s \|_C \, ds + M_4 \int_{t_0}^{t} \| \beta_s - \alpha_s \|_C \, ds$$

$$+ M_1 M_5 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds + M_2 M_5 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

$$= M_6 \int_{t_0}^{t} \| \beta_s - \alpha_s \|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C d\tau \, ds$$

where $M_6 = \max \{M_3 + M_4, (M_1 + M_2) M_5\}$. Thus $t \in I$, we have

$$(3.4) \quad \| \beta(t) - \alpha(t) \| \le M_6 \int_{t_0}^{t} \| \beta_s - \alpha_s \|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

Case 1.

Suppose $t \ge t_0 + r$. Then for all $\theta \in [-r, 0]$, $t + \theta \ge t_0$. For such θ's from (3.4), we get

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$$\|\beta(t+\theta) - \alpha(t+\theta)\| \le M_6 \int_{t_0}^{t+\theta} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t+\theta} \int_{t_0}^{s} \|\beta_\tau - \alpha_\tau\|_C \, d\tau \, ds$$

$$\le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \|\beta_\tau - \alpha_\tau\|_C \, d\tau \, ds$$

which yields

(3.5) $$\sup_{\theta \in [-r, 0]} \|\beta(t+\theta) - \alpha(t+\theta)\| = \|\beta_t - \alpha_t\|_C$$
$$\le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \|\beta_\tau - \alpha_\tau\|_C \, d\tau \, ds$$

Case 2.

Suppose $t_0 \le t < t_0 + r$. Then for all $\theta \in [-r, t_0 - t]$, we have $t + \theta < t_0$. we observe that

(3.6) $$\|\beta(t + \theta) - \alpha(t + \theta)\| = 0$$

For $\theta \in [t_0 - t, 0]$, $t + \theta \ge t_0$. Then, we get as in the case 1,

(3.7) $$\|\beta(t + \theta) - \alpha(t + \theta)\|$$
$$\le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \|\beta_\tau - \alpha_\tau\|_C \, d\tau \, ds$$

Thus, for every $\theta \in [-r, 0]$, $(t_0 \le t < t_0 + r)$, from (3.6) and (3.7), have

$$\|\beta(t + \theta) - \alpha(t + \theta)\| \le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \|\beta_\tau - \alpha_\tau\|_C \, d\tau \, ds$$

which yields

(3.8) $$\|\beta_t - \alpha_t\|_C \le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \|\beta_\tau - \alpha_\tau\|_C \, d\tau \, ds$$

For every $t \in I$, from (3.5) and (3.8), we have

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$$(3.9)\quad \| \beta_t - \alpha_t \|_C \le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C d\tau \, ds$$

$$\le \varepsilon + M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C d\tau \, ds$$

where $\varepsilon > 0$ is an arbitrary constant. Now, by applying Lemma 2 with $u(t) = \| \beta_t - \alpha_t \|_C$ we get from (3.9)

$$\| \beta_t - \alpha_t \|_C \le \varepsilon \, [1 + M_6 \exp\{(M_6 + 1)T\} T]$$

Since ε is arbitrary, $\| \beta_t - \alpha_t \|_C = 0$, $t \in I$, which implies that $\| \beta(t) - \alpha(t) \| = 0$, $t \in I$ i.e. $\beta(t) = \alpha(t)$, $t \in I$. Hence,

$$\beta(t) = \alpha(t), \; t \in J$$

This complete the proof of the Theorem 2.

PROOFS OF THEOREMS 3 AND 4

The solution of the system (1.1) on $(t_0 - r, \infty)$ is given by

$$x(t) = \phi(0) + \int_{t_0}^{t} f(s, x(s), x_s, \int_{t_0}^{s} k(s, \tau, x(\tau), x_\tau) \, d\tau) \, ds \qquad \text{for } t \ge t_0$$

(4.1)

$$= \phi(t - t_0), \qquad \text{for } t_0 - r \le t \le t_0$$

If $t \ge t_0$, then from (4.1) and using (2.3), (2.4), we get

$$(4.2)\quad \|x(t)\| \le \|\phi\|_C + \int_{t_0}^{t} L_2(s)[\| x\| + \| x_s\|_C] \, ds$$

$$+ \int_{t_0}^{t} L_2(s) \int_{t_0}^{s} L_1(\tau) \, [\| x\| + \| x_\tau\|_C \,] \, d\tau \, ds$$

$$\le \|\phi\|_c + \int_{t_0}^{t} 2L_2(s) \, \| x_s\|_c \, ds + \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau) \, \| x_\tau\|_C \, d\tau \, ds$$

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By proceeding as in the proof of Theorem 2, we get from (4.2)

$$(4.3)\ \|x_t\|_C \le \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\ \|x_s\|_C\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau)\ \|x_\tau\|_C\, d\tau\, ds$$

Applying Lemma 2 with $u(t) = \|x_t\|_C$, we obtain from (4.3),

$$(4.4)\ \|x_t\|_C \le \|\phi\|_C + [\,1 + \int_{t_0}^{t} 2L_2(s)\exp\{\int_{t_0}^{s} (2L_2(\tau) + L_1(\tau))\, d\tau\}\, ds$$

From (4.4) and (2.5), If follows that the solution $x(t)$ of the system (1.1) on $[t_o, \infty]$ is bounded. Consequently, the solution $x(t)$ of the system (1.1) is bounded on $[t_0 - r, \infty]$ and the proof of the Theorem 3 is complete.

Remark. It is to observe that our result proved in Theorem 3 yields the stability of the solution $x(t)$ of (1.1) if $\|\phi\|_C$ is small enough.

If $t \ge t_0$ then from (4.1) and making use of (2.6), (2.7), we obtain

$$(4.5)\ \|x(t)\| \le \|\phi\|_C + \int_{t_0}^{t} e^{-\gamma s} L_2(s)[\|x\| + \|x_s\|_C]\, ds$$

$$+ \int_{t_0}^{t} e^{-\gamma s} L_2(s) \int_{t_0}^{s} e^{-\gamma(\tau - s)} L_1(\tau)\ [\|x\| + \|x_\tau\|_C]\, d\tau\, ds$$

$$\le \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\, e^{-\gamma s}\, \|x_s\|_C\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau)\, e^{-\gamma t}\ \|x_\tau\|_C\, d\tau\, ds$$

From (4.5) and by proceeding as in the proof of Theorem 2, we get

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$$(4.6)\ \|x_t\|_C \le \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\, e^{-\gamma s}\, \|x_s\|_C\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau)\, e^{-\gamma\tau} + \|x_\tau\|_C\, d\tau\, ds$$

Now, since $\gamma > 0$ and $t \ge t_0 \ge 0$, we have $e^{-\gamma t}$ 5 $e^{-\gamma t}$5 1. Multiplying L.H.S. of (4.6) by $e^{-\gamma t}$ and R.H.S. of (4.6) by $e^{-\gamma t0}$, we obtain

$$(4.7)\ e^{-\gamma t}\|x_t\|_c \le \|\phi\|_C\, e^{-\gamma t_0} + \int_{t_0}^{t} 2L_2(s)\, e^{-\gamma t_0}\, e^{-\gamma s}\, \|x_s\|_c\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) e^{-\gamma t0} \int_{t_0}^{s} L_1(\tau) e^{-\gamma\tau}\, \|x_\tau\|_C\, dr\, ds$$

Applying Lemma 2 with $u(t) = e^{-\gamma t}\|x_t\|_C$, the inequality (4.7) yields

$$(4.8)\ \|x_t\|_C \le \|\phi\|_C \left[1+\int_{t_0}^{t} 2L_2(s) e^{-\gamma t_0} \exp\left(\int_{t_0}^{s} \{2L_2(\tau)e^{-\gamma t_0} + L_1(\tau)\}\, d\tau\right) ds\right] \cdot e^{l(t-t_0)}$$

From (4.8) and (2.5), it follows that

$$\|x_t\|_C \ge Q\, \|\phi\|_C\, e^{\gamma(t-t)}, \qquad t \le t_0$$

where $Q > 0$ is a constant. Hence, the solution $x(t)$ of system (1.1) is uniformly slowly growing and the proof of the Theorem 4 is complete.

ACKNOWLEDGEMENT

The author expresses his sincere gratidue to Professor B.G. Pachpatte for many helpful discussions and suggestions during the preparation of this paper. This research work is supported by Marathwada University, Aurangabad (Maharashtra) INDIA.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp-139-142

ON ORTHOGONAL IDEALS IN GROUPRINGS

W.B. Vasantha Kandasamy* & N. Suresh Babu*

(Received 15.9.92)

ABSTRACT

In this paper we define orthogonal ideals of a groupring and we prove when the ring is Z_p, p - a prime and G - a cyclic group of order n, there exists a pair of orthogonal ideals I_1 and I_2 in Z_pG such that $Z_pG = I_1 \cup I_2 \cup S$ where S is a semigroup under multiplication. Further we prove that these ideals are cyclic codes of dimension n-1 and 1 respectively.

DEFINITION 1

Let R be a ring. We say two ideals I and J are orthogonal if $I.J = O$ where $I.J = \{ i.j : i \in I, j \in J \}$.

The same definition holds for any grouping. Trivially every ideal is orthogonal to the (O) ideal.

Proposition 2

If R is a ring in which $R^2 = 0$, then every pair of ideals of R is orthogonal.

Proof: Obvious.

Proposition 3

If R is a ring in which $R^2 = 0$ and G is any group then in the grouping RG every pair of ideals is orthogonal.

Proof

As $R^2 = 0$ so $1 \notin R$. We have $RG.RG = (RG)^2 = RG = 0$. Hence by proposition 2, every pair of ideals in RG is orthogonal.

Example 1

$Z_2 = (0,1)$ and $G = \{ g : g^2 = 1 \}$. Then $Z_2G = (0, 1, g, 1+g)$. Let $I_1 = (0, 1+g)$, $I_2 = (0, 1+g)$,

$S = (0,1,g)$. Here I_1 and I_2 are orthogonal ideals and S is a semigroup under

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multiplication.Further

$Z_2G = I_1 \cup I_2 \cup S$

Example 2

Let $Z_2 = \{0,1\}$, $G = \{ g : g^3 = 1\}$ Then
$Z_2G = \{0, 1, g, g^2, 1+g, 1+g^2, g+g^2, 1+g+g^2\}$
Let $I_1 = \{0, 1+g, 1+g^2, g+g^2\}$, $I_2 = \{0,1+g+g_2\}$
Clearly $I_1.I_2 = 0$. Further $Z_2G = I_1 \cup I_2 \cup S$ where
$S = \{0, 1, g, g^2, 1+g+g^2\}$ is a semigroup under multiplication.

Example 3

$Z_2 = \{0,1\}$, $G = \{g : g^4=1\}$
$Z_2G = \{0,1, g, g^2, g^3, 1+g, 1+g^2, 1+g^3, g+g^2, 1+g+g^2, g+g^3, 1+g+g^3, 1+g^2+g^3, g+g^2+g^3, 1+g+g^2+g^3\}$
Let $I_1 = \{ 0,1+g, 1+g^2, 1+g^3, g+g^2, g+g^3, g^2+g^3, 1+g+g^2+g^3 \}$
$I_2 = \{ 0, 1+g+g^2+g^3 \}$

Clearly I_1 and I_2 are orthogonal. Now $Z_2G = I_1 \cup I_2 \cup S$ where
$S = \{0,1, g, g^2, g^3, 1+g+g^2, 1+g+g^3, 1+g^2+g^3, g+g^2+g^3 \}$ is a semigroup under multiplication.

Example 4:

$Z_3 = \{ 0, 1, 2 \}$, $G = \{ g : g^2 = 1\}$
$Z_3G = \{ 0, 1, 2, g, 2g, 1+g, 2+g, 1+2g, 2+2g\} = I_1 \cup I_2 \cup S$, where $I_1 = \{ 0, 2+g, 2g +1\}$, $I_2 = \{0, 1+g, 2+2g\}$ and $S = \{0, 1, 2, g, 2g, 1+g, 2+2g\}$

Theorem 1:

Let $Z_p = \{ 0, 1, '2, ...,p\text{-}1\}$ and $G = \{ g : g^n = 1\}$.
Then in the groupring Z_pG there exists a pair of orthogonal ideals I_1 and I_2 such that $Z_pG = I_1 \cup I_2 \cup S$ where S is a semigroup under multiplication. Further $I_1 \cap S = \{0\}$. $I_2 \cap S = \{0\}$ if p|n and $I_2 \subset S$ if p ∤ n.

Proof

$$Z_pG := \{\alpha = a_0 + a_1 g + \ldots + a_{n-1} g^{n-1} : a_i \in Z_p\}$$

Let $I_1 = \left\{ \alpha \in Z_pG : \sum_{1}^{n-1} a_i = 0 (mod\ p) \right\}$

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$I_2 = \{ k (g^{n-1} + g^{n-2} + \ldots + g + 1), 0 \leq k \leq p-1\}$ and

$S = \{0\} \cup \left\{ \alpha \in Z_p G : \sum_{i=1}^{n-1} a_i = 1 \text{ or } 2 \text{ or} \ldots \text{ or } (p-1) \pmod p \right\}$.

Clearly I_1 and I_2 are ideals in Z_pG. Now suppose $\alpha \in I_1$ and $\beta \in I_2$.

Then $\alpha = a_0 + a_1 g + \ldots + a_{n-1} g^{n-1}$, $\sum a_i = 0 \pmod p$.

$\beta = k (g^{n-1} + g^{n-2} + \ldots + g+1)$, $0 \leq k \leq p-1$.

Then $\alpha.\beta = (a_0 + a_1 + \ldots + a_{n-1}). (k (g^{n-1} + g^{n-2} + \ldots + g+1))$

$= \sum a_i . (k (g^{n-1} + g^{n-2} + \ldots + g+1))$

$= 0$, since $\sum a_i = 0 \pmod p$.

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Hence $I_1.I_2 = \{0\}$. Therefore I_1 and I_2 are orthogonal.

Clearly S is a semigroup under multiplication. Also $Z_pG = I_1 \cup I_2 \cup S$. By the definition of I_1 and S we have

$I_1 \cap S = \{0\}$. Now if $p|n$, then $n = mp$, for some positive integer 'm'. Then $I_2 = \{k(1 + g + \ldots + g^{mp-1}, 0 \leq k \leq p-1\}$. If $\alpha \in I_2$, we have $\sum a_i = k.. mp = 0 \pmod p$. Therefore, $I_2 \subseteq I_1$. Hence $I_2 \cap S = \{0\}$.

If $p \nmid n$, then $p \nmid k.n$ for any k such that $1 \leq k \leq p-1$. Therefore, $\sum a_i \neq 0 \pmod p$ for any $\alpha \in I_2$. That is, $\sum a_i = 1$ or 2 or ... or $(p-1) \pmod p$. Thus $\alpha \in S$ which implies $I_2 \subseteq S$.

Theorem 2:

In Z_pG, where $G = \{ g : g^n = 1\}$, the ideal defined by $I_1 = \{ \sum_{i=0}^{n-1} a_i g^i : \sum a_i = 0 \pmod p \}$ is a cyclic code generated by "$g-1$".

Recall that a cyclic code of length n and dimension k is a principal ideal generated by a divisor of degree $n-k$ of the polynomial g^n-1 in Z_pG.

Proof

Suppose $a_0 + a_1 g + \ldots + a_{n-1} g^{n-1} \in Z_pG$. Then we have,

$(g-1).(a_0 + a_1 g + \ldots + a_{n-1} g^{n-1})$

$= a_0 g + a_1 g^2 + \ldots + a_{n-1} g^n - a_0 - a_1 g - \ldots - a_{n-1} g^{n-1}$

$= (a_{n-1} - a_0) + (a_0 - a_1) g + \ldots + (a_{n-2} - a_{n-1}) g^{n-1}$.

Here $\Sigma a_i = 0$. Hence the above element belongs to I_1.

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Conversely, suppose $\sum_{i=0}^{n-1} b_i g^i \in I_1$. This implies that

$$\sum_{i=0}^{n-1} b_i = 0 \ (mod\, p)$$

We will find out an element $\sum_{i=0}^{n-2} a_i g^i$ of $Z_p G$ such that

$$\sum_{i=0}^{n-1} b_i g^i = (g-1).(\sum_{i=0}^{n-2} a_i g^i)$$

For, consider the element $\sum_{i=0}^{n-2} a_i g^i$ where a_i's are given by

$a_0 = -b_0$, $a_1 = -(b_0 + b_1)$, ...,$a_{n-3} = -(b_0 + b_1 + b_{n-3})$, $a_{n-2} = b_{n-1}$

Now, $(g-1).(\sum_{i=0}^{n-2} a_i g^i) = (g-1).\{-b_0 - (b_0 + b_1)g - \ldots + b_{n-1} g^{n-2}\}$.

$= b_0 + b_1 g + \ldots + b_{n-2} g^{n-2} + b_{n-1} g^{n-1}$,

since $\Sigma b_i = 0$ (mode p).

$$= \sum_{i=0}^{n-1} b_i g^i$$

Hence every element of I_1 is a multiple of $(g-1)$ and every multiple of $(g-1)$ belongs to I_1. This implies that I_1 is a cyclic code generated by $(g-1)$ and has dimension $n-1$.

Note : The ideal I_2 generated by $(g^{n-1} + g^{n-2} + \ldots + g + 1)$ is clearly the cyclic code of dimension 1.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 143-146

ON TRILATERAL GENERATING FUNCTIONS OF MODIFIED LAGUERRE POLYNOMIALS

Asit Baran Majumdar*

(Received 14-11-1992)

ABSTRACT:

In this paper we derive a theorem on mixed trilateral generating functions of modified Laguerre polynomials by group theoretic method. A particular case of interest is also pointed out.

AMS classification code (1980) : 33A65.

Keywords and Phrases: Generating functions, Modified Laguerre Polynomials.

INTRODUCTION.

The modified Laguerre polynomial $f_n^{\beta}(x)$ is defined by [1]

$$(1.1) \qquad f_n^{\beta}(x) = (-1)^n L_n^{-n-\beta}(x) = \frac{(\beta)_n}{n!} {}_1F_1\left[\begin{matrix} -n; \\ 1-\beta-n; \end{matrix} x\right]$$

In this note we prove the following theorem in connection with the trilateral generating relations of modified Laguerre polynomials from the Lie group view point.

Theorem : If there exists a bilateral generating relation of the form

$$(1.2) \qquad G(x,u,t) = \sum_{n=0}^{\infty} a_n f_{n-k}^{\beta}(x)\, g_n(u)\, t^n$$

where k is a non-negative integer and $g_n(u)$ is an arbitrary polynomial of degree n, then

$$(1.3) \qquad \sum_{n=0}^{\infty} \psi_n(x,u,t)\, z^n = (1-z)^{\beta-1} \exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right)$$

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where

$$(1.4)\qquad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n+k}{p+k} a_p (-1)^{n-p} f_{n+k}^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

PROOF OF THE THEOREM

We shall first define the following linear partial differential operator for modified Laguerre polynomials.

$$(2.1)\qquad R = xyz^{-1}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} + yz^{-1}(1-x)$$

such that

$$(2.2)\qquad R\left(f_{n+k}^{\beta}(x)\, y^n z^{\beta}\right) = -(n+k+1) f_{n+k+1}^{\beta-1}(x)\, y^{n+1} z^{\beta-1}.$$

The extended form of the group generated by R is given by

$$(2.3)\qquad e^{wR} f(x,y,z) = \left(\frac{z}{z-wy}\right) exp\left(\frac{wxy}{z-wy}\right) f\left(\frac{wz}{z-wy}, y, z-wy\right)$$

Consider the following formula

$$(2.4)\qquad G(x,u,t) = \sum_{n=0}^{\infty} a_n f_{n+k}^{\beta}(x)\, g_n(u)\, t^n$$

where $g_n(u)$ is an arbitrary polynomial of degree n.
Replacing t by tz and then multiplying both sides of (2.4) by y^{β}, we get

$$(2.5)\qquad y^{\beta} G(x,u,tz) = \sum_{n=0}^{\infty} a_n \left(f_{n+k}^{\beta}(x)\, z^n y^{\beta}\right) g_n(u)\, t^n$$

Operating both sides of (2.5) by $exp(wR)$, we get.

$$(2.6)\qquad exp(wR)\left(y^{\beta} G(x,u,tz)\right)$$

$$= exp(wR) \sum_{n=0}^{\infty} a_n f_{n+k}^{\beta}(x)\, y^{\beta}\, g_n(u)\, (tz)^n$$

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The left member of (2.6) becomes

$$(2.7)\qquad exp\left(\frac{-\frac{wxz}{y}}{1-\frac{wz}{y}}\right) y^{\beta}\left(1-\frac{wz}{y}\right)^{\beta-1} G\left(\frac{x}{1-\frac{wz}{y}}, u, tz\right)$$

Also the right member of (2.6) becomes

$$(2.8)\qquad \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^p}{p!} R^p\left(f_{n=k}^{\beta}(x)\, y^{\beta} z^n\right) g_n(u)\, t^n$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(-w)^p}{p!} (n+k+1)_p\, f_{n+k+p}^{\beta-p}(x)\, z^{n+p}\, y^{\beta-p}\, g_n(u)\, t^n$$

$$= \sum_{p=0}^{\infty} z^n \sum_{p=0}^{\infty} \binom{n+k}{p} a_{n-p} \left(-\frac{w}{y}\right)^p f_{n+k}^{\beta-p}(x)\, y^{\beta}\, g_{n-p}(u)\, t^{n-p}$$

Then equating (2.7) and (2.8) we get

$$(2.9)\qquad \left(1-\frac{wz}{y}\right)^{\beta-1} exp\left(\frac{-\frac{wxz}{y}}{1-\frac{wz}{y}}\right) G\left(\frac{x}{1-\frac{wz}{y}}, u, tz\right)$$

$$= \sum_{n=0}^{\infty} \psi_n(x,u,t)\, z^n$$

where

$$(2.10)\qquad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n+k}{p+k} a_p \left(-\frac{w}{y}\right)^{n-p} f_{n-k}^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

putting w/y =1 in (2.9), we get

$$(2.11)\qquad (1-z)^{\beta-1} exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right) = \sum_{n=0}^{\infty} \psi_n(x,u,t)\, z^n$$

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where

$$(2.12)\quad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n+k}{p+k} a_p (-1)^{n-p} f_{n-k}^{\beta,(n-p)}(x)\, g_p(u)\, t^p$$

This completes the proof of the theorem.

Putting $k = 0$, we get the following result of modified Laguerre polynomeals:

Corollary: If there exists a bilateral generating relation of the form

$$(2.13)\quad G(x,y,t) = \sum_{n=0}^{\infty} a_n f_n^{\beta}(x)\, g_n(u)\, t^n$$

then

$$(2.14)\quad (1-z)^{\beta-1} \exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right) = \sum_{n=0}^{\infty} \psi_n(x,u,t)\, z^n$$

$$(2.15)\quad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n}{p} a_p (-1)^{n-p} f_n^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

ACKNOWLEDGEMENT

The anthor is grateful to Dr. A.K. CHONGDAR, Reader, Department of Mathematics, Bangabasi Evening College, Calcutta for his kind help and gridance in the procpration of this paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp-147-152

ROBUST ROW-COLUMN DESIGNS WITH RANDOM ROW AND COLUMN EFFECTS

Gulab Singh*

(Received : 30-03-1991, after revision: 23-12-1992)

ABSTRACT

Gopalan and Dey [4] have developed a criterion, on the lines similar to one suggested by Box and Draper [2], for characterising the robust experimental designs assuming a fixed effect model or model - I (Eisenhart [3]). In this paper a criterion for characterising robust experimental row-column designs assuming random row and column effects has been developed and the robustness of the row column designs with random row and column effects has been examined.

Mathematics Subject Classification (1991) : *62K99*

Keywords : Random effect model, robustness, idempotent matrices, eigen values, outlier, equi-replicate, Kronecker product of matrices, heterogeneity, dispersion matrix, normal equations, incidence matrix.

INTRODUCTION

It is assumed that the analysis of variance model for the observations can be approximated by linear combinations (functions) of certain unobservable quantities known as 'effects'. The 'effects' not directly observable quantities are more of less idealised formulations of some properties of interest to the investigator in the phenomenon underlying the observations. The purpose of the analysis is to make inferences about some of the effects and these inferences to be valid regardless of the magnitudes of certain other effects, which may be present in the linear combinations. Each affect is regarded as either an unknown constant or else as a random variable. Following Eisenhart [3], if the effect is treated as an unknown constant, it is called a fixed effect model or model -I, otherwise it is called a random effect model or model - II. The equation expressing the observations as linear combinations of the effects is known as model equation. It is assumed that in every model equation there is at least one set of random effects equal in number to the number of observations, a different one

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of which appears in every observation. This is called the residual effect or the error term. Furthermore, there is usually one fixed effect, which appears in every model equation and is known as the general constant and is the mean in some sense of the observations.

A number of criteria have been enumerated by Box [1] for judging a good response surface design for fitting an empirical interpolation function $Y = X\beta + e$ where $Y = (y_1, y_2, \ldots, y_n)'$ is an nx 1 vector of response observations, X is an nxp matrix of predictor variable observations, β is a px1 vector of parameters to be estimated and e is an nx 1 vector of residuals. On of the criterion is that the design should be insensitive to the wild observations. Box and Draper [2] have derived an appropriate numerical measure of a design's desirability in relation to the insensitivity to wild observations. Box and Draper [2] have shown that in order that the predicted response y at any given point be insensitive to the outlier, the quantity $r = \Sigma r_{uu}^{\ 2}$ should be minimised, where r_{uu} is the u-th diagonal element of the matrix $R = X(X'X)^{-1}X$. The designs which minimise r are called *robust*.

2. ROW-COLUMN DESIGNS WITH RANDOM ROW AND COLUMN EFFECTS

We shall consider a row-column design in which v treatments are arranged in p rows and q columns, i-th treatment being replicated r_i times. The model assumed in this case will be

$$Y = \mu J + \pi \tau + L \phi + M \Phi + e$$

where $y = ((y_{ijk}))$ is the vector of response of the i-th treatment in j-th row and k-th column, μ is the general mean effect, $\tau = (\tau_1, \ldots, \tau_v)'$ is the vector of treatment effects, $\phi = (\phi_1 \ldots, \phi_p)$ is the vector of row effects, $\Phi = (\phi, \ldots, \phi_q)'$ is the vector of column effects, e is the vector of random errors, independently normally distributed with mean vector O and dispersion $\sigma^2 I$. Here ϕ and Φ are assumed to be vector of random effects distributed normally with mean vectors O and dispersion matrices $\sigma_r^2 I$ and $\sigma_c^2 I$ respectively. π', L' and M' are the incidence matrices of treatments versus observations, row versus observations and column versus observations respectively. I is an identity matrix and J is a vector with all elements as unity. The above model may be written as

$$Y = X\ 0 + e_1$$

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where $e_1 = L\phi + M\phi + e$, $X = [J : \pi_{n,v}]$ and $\theta = (\mu, \tau)'$ we have the dispersion matrix of e_1 as,

$$D(e_1) = LL'\ \sigma_r^2 + MM'\ \sigma_c^2 + I\ \sigma^2$$
$$= Z^{-1}\ \sigma^2 \quad \text{(say)}$$

where Z^{-1} is the variance-covariance matrix of the random components and is function of σ_r^2/σ^2 and σ_C^2/σ^2 which are assumed to be fixed and known. Application of usual least square technique to estimate the vector of parameters yields the normal equation,

$$\hat{\theta} = (X'\ Z^{-1} X)^{-} X\ Z^{-1} Y$$

where $(X'Z^{-1}X)^{-}$ is generalised inverse (g - inverse) of $X'Z^{-1}X$ satisfying $X'Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1}X = X'Z^{-1}X$. It is well known that in the absence of any outlier $R_0^2/(n-m)$ is an unbiased estimator of σ^2 where,

$$R_0^2 = Y'\ [Z^{-1} - Z^{-1}X\ (X'Z^{-1}X)^{-}X'Z^{-1}]Y$$

m = rank (X) and n is the number of observations.

When u-th observation has added to it a quantity c making it an outlier, the bias or discrepancy in estimating σ^2 through $R^{*}_{0}{}^{2}$ is $d_u = c^2 a_{uu}/(n-m)$ where $R^{*}_{0}{}^{2}$ is the residual sum of squares in the presence of the outlier and a_{uu} is the u-th diagonal element of the matrix

$$A = Z^{-1} - Z^{-1}X\ (X'Z^{-1}X)^{-}X'Z^{-1}$$
$$= Z^{-1} - D\ (Z^{-1}X'\ \hat{\theta})$$

where D stands for the dispersion matrix. If it were equally likely that c could occur with any of the n observations, giving rise to $d_1,\ldots,d_n$, the average discrepancy would be

$$\bar{d} = c^2 \sum_u a_{uu} / n(n-m)$$

which obviously is not fixed, unlike in the case of the fixed effect model, but is design dependent through A. In order that no unduly large discrepancy in the estimator of σ^2 is caused by the outlier, the average discrepancy should be minimised. A design minimising the average discrepancy, in case of a random effect model, may be called *robust*.

We have,

$$D(Z^{-1}X'\hat{\theta})/\sigma^2 = Z^{-1}\pi'\ (\pi Z^{-1}\pi')^{-1}\pi' Z^{-1}$$

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For getting the average discrepancy we need

$$\begin{aligned} trace\ Z^{-1}\pi'\,(\pi Z^{-1}\pi')^{-1}\pi Z^{-1} &= trace\,(\pi Z^{-1}\pi')^{-1}\pi Z^{-1}Z^{-1}\pi' \\ &= trace\,(\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi' \end{aligned}$$

We examine below, robustness of row-column designs based on the criterion discussed above.

3. ROBUSTNESS OF ROW-COLUMN DESIGN

In case of a row-column design in v treatments, p rows and q columns the form of the matrix Z will be,

$$\mathbf{Z} = \begin{bmatrix} (1+\delta_2)+\delta_1 J & \delta_2 I & \cdots & \delta_2 I \\ \delta_2 I & (1+\delta_2)+\delta_1 J & \cdots & \delta_2 I \\ \vdots & \vdots & & \\ \delta_2 I & \delta_2 I & \cdots & (1+\delta_2)+\delta_1 J \end{bmatrix}$$

or $Z = I_p \otimes [I_q + \delta_1 J_q] + J_p \otimes \delta_2\ I_q$

$$= I_p \otimes I_q + \delta_1 I_p \otimes J_q + \delta_2 J_p \otimes I_q$$

where $A \otimes B$ denotes the Kronekcer product of matrices A and B. J_n is an nxn matrix with all elements as unity, $\delta_1 = \sigma_r^2/\sigma^2$ and $\delta_2 = \sigma_c^2/\sigma^2$. The form of Z^{-1} will be

$$Z^{-1} = a_{oo}\, I_p \otimes I_q + a_{01}\, I_p \otimes J_q + a_{10}\, J_p \otimes I_q + a_{11}\, J_p \otimes J_q$$

where ,

$$a_{oo} = 1$$
$$a_{01} = -\delta_1/(1+q\delta_1)$$
$$a_{10} = -\delta_2/(1+p\delta_2)$$
$$a_{11} = -\delta_1\delta_2(2+p\delta_2+q\delta_1)/[(1+p\delta_2)(1+q\delta_1)(1+p\delta_2+q\delta_1)$$

In order to compute the average discrepancy in estimating σ^2 through R^{*2}_0 in the presence of an outlier, we need trace $(\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi'$. For a row-column design we have,

$$\pi Z^{-1}\pi' = \pi[I_p \otimes I_q + a_{01} I_p \otimes J_q + a_{10} J_p \otimes I_q + a_{11} J_p \otimes J_q]\,\pi'$$
$$= r^d + a_{01} N_1 N_1' + a_{10} N_2 N_2' + a_{11} rr'$$

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and

$$\pi Z^{-2}\pi' = r^d + b_{01} N_2N_1' + b_{10}N_2N_2' + b_{11}rr'$$

where

$$b_{01} = -\delta_1 (2 + q\delta_1) / (1 + q\delta_1)^2$$

$$b_{10} = -\delta_2 (2 + p\delta_1) / (1+p\delta_2)^2$$

$$b_{11} = 2 (1 + p\delta_2 + q\delta_1) - pq(p\delta_2 + q\delta_1)\ \delta_1\delta_2 / [(1+p\delta_2 + q\delta_1)(1+p\delta_2)(1+ q\delta_1)]$$

and r^d is a diagonal matrix with elements of the replication vector r in the diagonal elements, N_1 and N_2 are the treatment versus rows and treatment versus column incidence matrices respectively.

We shall consider binary equi-replicate row-column designs for which N_1N_1' and N_2N_2' commute. For such designs

$$\text{trace}\ \left(\pi Z^{-1}\pi'\right)^{-1}\left(\pi Z^{2}\pi'\right) = \frac{r \cdot b_{01}vr + b_{10}vr + b_{11}vr^2}{r + a_{01}vr + a_{10}vr + a_{11}vr^2} + \sum_{i=1}^{v-1} \frac{r + b_{01}\theta_{1i} + b_{10}\theta_{2i}}{r + a_{01}\theta_{1i} + a_{10}\theta_{2i}}$$

where θ_{1i} and θ_{2i} are the *i-th* eigen values of the matrices N_1N_1' and N_2N_2' respectively. From equation (3.1) it is clear that the average discrepancy is not fixed, as in the case of fixed effect model, but is design dependent through θ_i's. In order that the average discrepancy is minimum, the trace given in (3.1) should be maximum subject to the condition that sum of all the eign values is constant. So maximising (3.1) subject to the condition that $\sum_{i=1}^{v-1}\theta_{1i} = r(v-k)$ and $\sum_{i=1}^{v-1}\theta_{2i} = r(v-k)$, we may see that all θ_{1i}'s are same and so are θ_{2i}s. Let $\theta_{1i} = \theta_1$ and $\theta_{2i} = \theta_2$ so that $\theta_1 = r(v-q) / (v-1)$ and $\theta_2 = r(v-p) / (v-1)$

The characteristic vector corresponding to the eigen value rq is $rq\ JJ'$ and the eigen vector corresponding to the eigen value $r(v-q)/(v-1)$ is $(I-JJ'/v)$, so we have,

$$N_1N_1' = rq\ JJ'/v + r(v-q)\ (v-1)^{-1}\ (I - JJ'/v)$$

similarly,
$$N_2N_2' = rp\ JJ'/v + r(v-p)\ (v-1)^{-1}\ (I - JJ'/v)$$

giving,

$$pN_1N_1' + qN_2N_2' = r\ (v-1)^{-1}\ [\{\ v\ (p + q) - 2pq\ \}\ I - \{p + q - 2pq\}\ JJ']$$

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which is satisfied by balanced incomplete row-column designs (Singh [5]), showing that balanced incomplete row-column designs with random row and column effects are robust.

ACKNOWLEDGEMENT

The author is grateful to the referee for his constructive comments on the earlier draft of the paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp-153-158

INTER BLOCK ANALYSIS OF BALANCED INCOMPLETE BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

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(Received: 25-12-1992, after revision: 30-03-1993)

ABSTRACT

Balanced incomplete block designs with nested rows and columns (BIBRC) were introduced by Singh and Dey [2] as generalisation of the Lattice Square designs. The intra-block analysis of these designs has been developed by Singh and Dey [2]. Since the blocks of the BIBRC are incomplete, assuming the block effects as random and row, column classification complete, an attempt has been made in this paper to develop the inter block analysis of these designs.

Mathematics Subject Classification (1991) : 62K10

Keywords : Lattice Square designs, heterogeneity, orthogonal, dispersion matrix, Kronecker product of matrices, maximum likelihood estimates, random effect model.

INTRODUCTION

The experimental units within a complete block are often subjected to more than one source of variation. Recognising the need for appropriate experimental designs for such situations Yates [3,4] developed a group of block designs with possibility of eliminating heterogeneity in two directions within the complete block. Singh and Dey [2] defined and studied a class of experimental designs which are arrangements of treatments in several sets (blocks) of rows and columns such that within each set (block) the row versus column classification is orthogonal, though the sets themselves are incomplete. These designs may be viewed as the generalization of the Lattice Square

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designs of Yates [4]. Designs defined by Singh and Dey [2] are different from the designs of Preece, Pearce and Kerr [1] who consider designs with three mutually orthogonal and fully crossed classifications while Singh and Dey [2] consider designs with a set of blocks (sets) within which are nested two more classifications, rows and columns.

NOTATIONS AND FORMULATION OF THE MODEL FOR ANALYSIS

Let us suppose that v treatments have been allotted to s blocks (sets) and the experimental units within each set are further arranged in p rows and q columns, enabling to control the variability within a set, in two directions. The model assumed in this case is

$$y_{i_{(j_m j_l)}} = \mu + \Omega_j + \phi_{j_m} + \delta_{j_l} + \tau_i + \varepsilon_{i_{(j_m j_l)}} \qquad (1)$$

where, $y_{i_{(j_m j_l)}}$ is the response obtained from the unit receiving i-th treatment in the $(j_m j_l)$-th cell of the j-th set, μ is the general mean effect, Ω_j is the effect of the j-th set (block) ($l = 1,.....,s$), ϕ_{j_m} is the effect of m-th row of the j-th set ($m = 1 ,....., p$), δ_{j_l} is the effect of the l-th column in the j-th set ($l = 1,...., q$), τ_i is the effect due to i-th treatment ($i = 1,...., v$) and $\varepsilon_{i_{(j_m j_l)}}$'s are uncorrelated residuals distributed normally with mean zero and constant variance σ^2. Since model (1) is assumed as random effect model, we further assume the following distributional properties of different effects i.e. δ_{j_l} is distributed normally with mean zero and variance $\sigma^2_{c_j}$, ϕ_{j_m} is distributed normally with mean zero and variance $\sigma^2_{r_j}$ and Ω_j is distributed normally with mean zero and variance σ_s^2. Also ϕ_{j_m}, δ_{j_l}, and $\varepsilon_{i_{(j_m j_l)}}$ are independently distributed. further,

$$V[y_{j_{(m,l)}}] = \sigma^2 + \sigma^2_s + \sigma^2_{r_j} + \sigma^2_{c_j}$$

$$Cov[y_{j_{(m,l)}}, y_{j_{(m',i)}}] = \sigma^2_s + \sigma^2_{c_j} \qquad , m \neq m'$$

$$Cov[y_{j_{(m,l)}}, y_{j_{(m,l')}}] = \sigma^2_s + \sigma^2_{r_j} \qquad , l \neq l'$$

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$$Cov[\, y_{j(m,l)}, y_{j(m',l')}\,] = \sigma^2 s \qquad , l \neq l', m \neq m'$$

$$Cov[\, y_{j(m,l)}, y_{j(m,l)}\,] = 0 \qquad , j \neq j'$$

Let $y_{j(m,.)}$, $y_{j(.,.)}$ and $y_{.(.,.)}$ denote the total of the l-th column of the j-th set, total of the j-th set and grand total respectively. Further let us denote the vectors of the column totals of the j-th set, totals of the set (block) and grand total respectively as,

$$Y_{j(m,.)} = [y_{j(m,1)}, \ldots\ldots\ldots, y_{j(m,q)}]'_{qx1}$$

$$Y_{j(.,.)} = [y_{j(1,.)}, \ldots\ldots\ldots\ldots, y'_{j(m,.)}]'_{pqx1}$$

$$Y_{(.,.)} = [y'_{1(.,.)}, \ldots\ldots\ldots\ldots, y'_{s(.,.)}]'_{spqx1}$$

it may easily be verified that

$$D\left[Y_{j(m,.)}\right] = (\sigma^2 + \sigma^2_{cj})I + (\sigma^2_s + \sigma^2_{rj})J_{q,q}$$

$$Cov\left[Y_{j(m,.)}, Y_{j(m',.)}\right] = \sigma^2_{cj} I + \sigma^2_s J_{q,q} \qquad m \neq m'$$

$$Cov\left[Y_{j(m,.)}, Y_{j'(m',.)}\right] = 0_{q,q} \qquad j \neq j',\ m \neq m'$$

where **D**(.) stands for the dispersion matrix, I is the identity matrix and J is the matrix with all elements as unity. Thus,

$$D[y_{j(m,.)}] = \sigma^2 I_{pq} + \sigma^2_{rj}\, I_p \otimes J_{q,q} + \sigma^2_{cj} J_{p,p} \otimes I_p + J_{qp,\, pq}$$

V_j *(say)* $j = 1, \ldots, s$

$$D[Y_{.(.,.)}] = diag(V_1, \ldots, V_s)$$

$$= I_s \otimes (\sigma^2 I_{pq} + \sigma_s{}^2 J_{pq,qp}) + diag\,(\sigma^2_{r_1}, \ldots, \sigma^2_{r_s}) \otimes (I_p \otimes J_{q\text{-}p})$$

$$+ diag\,(\sigma^2_{c_1}, \ldots, \sigma^2_{c_s}) \otimes (J_p \otimes I_p) \qquad (2)$$

$$= V \text{ (say)}$$

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Where $\otimes$ stands for Kronecker product of matrics. Under these assumptions, the joint distribution of the elements of the response vector y is given by

$$f(y, \theta) = [(2\pi)^{n/2} |V|^{1/2}]^{-1} \exp^{-1/2} (y-\mu J - \Delta'\tau)' V^{-1} (y- \mu J - \Delta'\tau)$$

Where $\theta' = (\mu, \sigma^2, \sigma^2_s, \sigma^2_{r_j}, \sigma^2_{c_j})$, $\Delta_{v,m}$ is treatment versus observation incidence matrix and $|V|$ is the determinant of V, From (2) we have,

$$V^{-1} = diag\ (V_1^{-1}, V_2^{-1}, ..., V_s^{-1})$$

with $$V_j^{-1} = a_j I_p \otimes I_q + b_j I_p \otimes J_p + c_j J_{p,p} \otimes I_q + d_j J_{p,p} \otimes J_{q,q}$$

where a_j, b_j, c_j and d_j are functions of σ^2, $\sigma_s^2 \sigma^2_{c_j}$ and $\sigma^2_{r_j}$ given by

$$a_j = 1/\sigma^2$$

$$b_j = -\rho^2_{r_j} /\{(1+q\rho^2_{r_j}) \sigma^2\}$$

$$c_j = -\rho^2_{c_j} /\{(1+q\rho^2_{c_j}) \sigma^2\}$$

$$d_j = - \frac{\rho^2_{r_j} \rho^2_{c_j} (2 + p\rho^2_{c_j} + p\rho^2_{r_j} + pq\rho^2_s) - \rho^2_s}{\sigma^2 (1 + p\rho^2_{c_j}) (1 + \rho^2_{r_j}) (1 + p\rho^2_{c_j} + q\rho^2_{r_j} + pq\rho^2_s)}$$

where $\rho^2_{r_j} = \sigma^2_{r_j}/\rho^2$, $\rho^2_{c_j} = \sigma^2_{c_j}/\sigma^2$ and $\rho_s^2 = \sigma_s^2/\sigma^2$

Writing model (1) as $y = X\beta$, where β is the vector of parameters, maximum likelihood estimates of the vector of parameters may be obtained by solving the normal equations.

$$\left(\sum_{j=1}^{s} X_j' V_j^{-1} X_j\right)\beta = \sum_{j=1}^{s} X_j' V_j^{-1} y_j \qquad (3)$$

The left hand side of equation (3) may be simplified further. Let α_{ij} denote the incidence vector of the i-th treatment in the j-th set. Obviously, we have

$$\alpha'_{ij}\alpha_{ij} = r_j$$
$$\alpha'_{ij}\alpha_{i'j} = 0 \qquad i \neq i' \text{ and}$$
$$J'\alpha_{ij} = r_{ij}$$

r_{ij} being the number of times the i-th treatment has been replicated in

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the j-th set, r_j is the vector of replication of treatments in the j-th set and $J=(1, 1,.....,1)'$, Model (1) may also be written as,

$y_i = \mu J + \tau +$ error components of sets, rows within set and columns within set and the residual

$$E(y_j) = \left[J_{pq,1} : \mathbf{a}_{ij}, \ldots, \mathbf{a}_{vj}\right] \begin{bmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_v \end{bmatrix}$$

$$= X_j \beta$$

Using these notations it may easily be verified that the term on the left hand side of equation (3) is

$$\left(\sum_{j=1}^{s} X'_j V^{-1}_j X_j\right) = \begin{bmatrix} J'V^{-1}J & \sum_{j=1}^{s} J'V_j^{-1}\alpha_{1j} & \sum_{j=1}^{s} J'V_j^{-1}\alpha_{vj} \\ & \sum_{j=1}^{s} \alpha'_{ij}V_j^{-1}\alpha_{ij} & \sum_{j=1}^{s} \alpha'_{ij}V_j^{-1}\alpha_{ij} \end{bmatrix}$$

The general solutionof the normal equation(3) is rather complicated. However, for a particular design and for various values of the ratios, ρ_{r_j}, ρ_{i_j} and ρ_s the inter block estimates of the parameters may be obtained by solving the aforesaid normal equations.

ACKNOWLEDGEMENTS

The author is grateful to Dr. A. Dey and Dr. M. Singh, Professor and Scientist S-1 respectively at Indian Agricultural Statistics Research Institute, New Delhi, for their guidance and suggestions during the preparation of this paper. The author is also thankful to the referee for his very helpful comments on the previous version of the paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 159-166

HINDU TRIGONOMETRY✝

S. L. Singh* & Ramesh Chand*

(Received 17.07.1993)

ABSTRACT

In this paper we attempt to give a historical back ground of trigonometry and some trigonometrical formulae form *siddhanta Siromaṇi*. The conclusion is that the trgonometrical functions travelled from India to Europe via Arab.

Mathematics Subject Classifications (1991). : 01A32

Keywords & phrases : Trigonometrical functions, Hindu sine/cosine/versed sine.

THE ORIGIN OF TRIGONOMETRICAL FUNCTIONS AND THEIR HINDU NAMES

JYĀ

The fundamental term "*jyā*" of trigonometry appears in *Ṛgveda* (6.75.3).

योषेव शिङ्क्ते वितताधि धन्वञ्ज्या इयं समने पारयन्ती ।
अहिरिव भोगैः पर्येति बाहुं ज्याया हेतिं परिबाधमानः ॥

"*Jyā*" is the original word of Sanskrit language. The literal meaning of *jya* is the string of the *dhanusa* (bow) (See also [10, p. 197) and [17, p. 24].

"*Jyā*" also appears in *Sūrya Siddhānta*. We quote the following (cf. [7], [9] and [14].

राशिलिप्ताष्टमो भागः प्रथमं ज्यार्धमुच्यते ॥
तत्तद्विभक्तलब्धोनमिश्रितं तद् द्वितीयकम् ॥
आद्येनैवं क्रमात् पिण्डान् भक्त्वा लब्धोनसंयुताः ॥
खण्डकाः स्युश्चतुर्विंशज्ज्यार्धपिण्डाःक्रमादमी ॥

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✝ Its major part was presented at National Seminar on Science in Ancient India Oct 12-16, 1991, Kumaun University, Nainital

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Indeed trigonometry forms the base of *Sūrya Siddhānta*, and it developed in India as the stern daughter of astronomy.

A special name for the function *jyā* which we call the sine also appears in the *Ganita - Pada* of *Āryabhaṭiya* of *Āryabhaṭa* (b. 476 A.D.). However instead of *jyā*, he used *ardhajyā* (half chord). A (Hindu) sine table has also been given by *Āryabhaṭa* (see [14, p. 49]).

The word *jya* also appears in *Pnacasiddhāntikā* of *Varahamihira* (b. 505 A.D.) (See [15]). He also formed a (Hindu) sine table with a diameter of 120.

The term *jyā* formed its way into the works of *Brahmagupta* as *kramajyā* i.e. a straight line or *sinus rectus*.

Now the question arises how this word *jyā* is changed to sine. When this word transferred from India to Arab, it was pronounced *jaib* (see Gingerich [4], Smith [13] and Srinivasiengar [14]). And Arabians used this unrelated word. In fact *jaib* means bosom, breast and bay. When Gherardo of Cremona (1150 A.D.) made his translation from the Arabian literature, he used *sinus* for *jaib*. *Sinus* means bosom, bay, a curve, the fold of the toga about the breast, the land about a gulf and a fold in a land.

The word *kramajyā (jyā)* changed into *karaja* and *kardaja* at the hands of Arabians.It appears in the Bagdad school in the 9th century. In particular, *Al-Khowārizmī* used in the extract, which he made from the *Brahmasphutasiddhanta* of *Brahmagupta*. This work is known *Sindhind*. This word *jyā* also appears in the writings of the Spanish *Arab-ibn-Al-Zargāla* (1050 A.D.).

ABBREVIATION OF *JYĀ* (SINE)

The first mathematician who abbreviated sine into sin was Girard (1626 A.D.), (cf. [13]).

KOTIJYĀ (COSING)

Cosine is the modern name of Sanskrit word *kotijyā* that was used by ancient Indian astronomers. It appears in *Sūrya Siddhānta* and in the works of *Āryabhaṭa*. *Koti* means the (perpendicular) side of a right triangle, but it also

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means the end of *dhanuṣa* (bow). In other words *koṭijyā* means complementary *jya*

ABBREVIATION FOR *KOṬIJYĀ* (COSINE)

Plato of Tivoli (1120 A.D.) used *chorda residui* or the complement angle .Regiomontanus (1463 A.D.) used *sinus rectus* complementi. Rhaeticus (1515 A.D.) preferred basis. Vieta (1579 A.D.) used *sinus residuae*. Magini (1609 A.D.) used *sinus secundes*. Edmud Gunter (1620 A.D.) suggested *co-sinus* John Newton (1658 A.D.) used *cosinus* and this word received general favour. Jonas Moore (1674 A.D.) used cos and this symbol generally has been adopted by later writers (cf. Smith [13]).

UTKRAMAJYĀ (VERSED SINE)

The function *utkramajyā* first appears in *Surya Siddhānta* (280-400 A.D.) and immediately in the writings of *Āryabhata* I. It is also known as *ujjayā*. *Bhāskarācārya* in his *Līlāvati* calls it *Śara* (see below). It is (now) denoted by *Hvers Sin θ (Hvers θ) = R - Hcos θ*, wherein R is the radius of the circle. In modern language it is denoted by *Versin θ* = 1-cosθ (see Smith [13, part II, p. 618] and Kumari [7, Part I, p. 117]) From India it passed to Arab and they called it turned chord that is the arc of a circle.

THE VALUES OF *JYĀ* AND *KOṬIJYĀ* FUNCTIONS

Munjāla (932 A.D.) defined the positive and negative values of the functions *jyā* and *Kotijyā* (see for instance, [16,p. 250]), in the following verse of his *Laghumānasa*

ग्रहः स्वोच्चोनितः केन्द्र तदूघ्वाधोऽर्धजो भुजः ।
धनर्ण पदशः कोटी घनर्णर्ण धनात्मिका ॥

This means :

In the upper semicircle (1-2 quadrants), the value of the function *jyā* remains positive and the function *kotijyā* remains +,-,-,+ in the four quadrants respectively.

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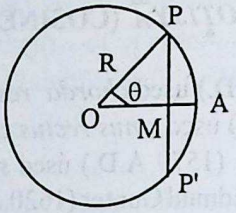
HINDU DEFINITION OF *JYĀ* (SINE), *KOṬIJYĀ* (COSINE) AND *UTKRAMAJYĀ* OR *S'ARA* (VESED SINE)



In figure 1, 0 is the centre of the circle, PP' is the full chord of the circular arc PAP' and A is the middle point of the arc PAP'. On account of its shape PAP' is called the *Cāpa* . Let the arc PA subtend an angle θ at the cente 0. In Hindu trigonometry the angle θ is connoted by the corresponding arc AP of the circle. The chord PP' is the string of the bow and is called *jyā* in Sanskrit. PM which is half of the full chord PP' is known as (*Ardhyajyā* or simply *jyā*). Thus

Jya $\overset{\frown}{PA}$ = PM = *Hsin* θ

Kotijya $\overset{\frown}{PA}$ = OM = *Hcos* θ

Versed sine $\overset{\frown}{PA}$ = MA = *Hvers* θ

where PA stands for the arc PA and H for Hindu. Thus

Hsin θ = *Rsin* θ, *Hcos* θ = *Rcos* θ and
Hvers θ = *Rvers* θ = *R-Hcos* θ (see [1],[2],[5] and [8]).

VALUE OF RADIUS R

The circumference of the circle is divided into 360 x 60 = 21600 equal parts and each part is taken to be a unit length, i.e. *Kalā* (minute).

Thus $2\pi R = 21600$
$R = 21600 \div 2\pi$, where $\pi = 3.1416$

The radius R is generally supposed to be 3438' (see [1], [2], [3], [7], [11], [12] and [16]. Recently, Hogendijk [6] has illustrated that R is half the circumference of the circle, and that

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$R = (180/\pi)^\circ = (10800/\pi)' = 3437.7' \cong 3438'$

He [op. cit.] further says, "Sine tables with the base $R = 3438'$ are of (pre-Islamic) Indian origin and were known in the early Islamic tradition, although no such tables survive in the Arabic sources".

Sometimes the radius is taken to be 120 units, and *Brahmagupta's Laghujyā* is indeed Hsine in this case. *Bhāskarācārya* also uses these *Laghujyas* in the rectification of planets. *Bhāskarācārya takes* the $R = 3437.4766$ units (see [1] which is evidently a slightly better than 3438 units.

5. SOME BASIC FORMULAE

It is obvious from the right triangle POM,

(1) $Hsin^2\theta + Hcos^2\theta = R^2$

In modern language,

$$R^2 sin^2\theta + R^2 Cos^2\theta = R^2$$

which gives the well - known formula of trigonometry :

$$sin^2\theta + cos^2\theta = 1$$

Bhāskarācārya writes in *Golādhyāya* ;

क्रमोत्क्रमज्याकृतियोगमूलाद्दलं तदर्धाशकशिञ्जिनी स्यात् ।
त्रिज्योत्क्रमज्यानिहतेर्दलस्य मूलं तदर्धाशकशिञ्जिनी वा ॥
तस्याः पुनस्तद्दलभागकानां कोटेश्च कोट्यंशदलस्य चैवम् ।
अन्यज्यकासाधनमुक्तमेवं पूर्वैः प्रवक्ष्येऽथ विशिष्टमस्मात् ॥

Thus according to Somayaji [2] means :

(2) $Hsin\ \theta/2 = \frac{1}{2}\ (Hsin^2\theta + Hvers^2\ \theta)^{1/2}$

(3) $Hsin\ \theta/2 = (\frac{1}{2}\ R \times Hvers\ \theta)^{1/2}$

We give a proof of these formulae,

Proof. In figure 2, let $\angle PMT = \theta$, PS $\perp$ OT, K the middle point of PT and R

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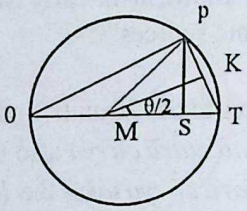
the radius of the circle OPT. Then

$$\text{KMT} = \frac{\theta}{2} \text{ and } \text{TK} = \frac{1}{2} \text{ PT}$$



The following steps are evident from the right triangle PST.

$$PT^2 = PS^2 + ST^2$$

$$PT^2 = Hsin^2\theta + Hvers^2\theta$$

$$\text{and } Hsin\frac{\theta}{2} = TK = \frac{1}{2}PT = \frac{1}{2}(Hsin^2\theta + Hvers^2\theta)^{1/2}$$

From the right triangle POT,

$$PT^2 = TS \times OT$$

$$PT^2 = (Hvers\ \theta) \times 2R \text{ (since OT} = 2R)$$

$$PT = (2R \times Hvers\ \theta)^{1/2}$$

$$2Hsin\frac{\theta}{2} = (2R \times Hvers\ \theta)^{1/2}$$

$$Hsin\frac{\theta}{2} = \frac{1}{2}(2R \times Hversa\ \theta)^{1/2} = (\frac{1}{2}R \times Hvers\theta)^{1/2}$$

Bhāskarācārya has given several other trigonometrical formulae. The verse numbers (12), (13), (14), (15), (16), (17), (18), (19), (20), (21) and (22) of *Golādhyāya* (*jyot-pattivāsanā*) give the following formulae :

(4) $$Hsin\frac{(90^\circ + x)}{2} = [1/2(R^2 + RHsin\ x)]^{1/2}$$

5) $$\sqrt{2}\ Hsin\left(\frac{x - y}{2}\right) = [1/2\{(Hsinx - Hsiny)^2 + (Hcosx - Hcosy)^2\}]^{1/2}$$

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(6) $[(H\cos x - H\sin x)^2]^{1/2} = H\sin(45° - x)$

(7) $R - (2H\sin^2 x)/R = H\sin(90° - 2x)$

(8) $H\sin(x \pm 1) = H\sin x\ (1 - 1/6569) \pm 10/573\ H\cos x$

(9) $H_{r+1} = H_r(1 - 1/476) + H\cos x_r\ (100/1529)$

(10) $H\sin(x + y)\ (H\sin x.\ H\cos y + H\cos x.H\sin y) \div R$

For details refer to Somayaji [1] and [2].

For the proof of the formula (8) Somayaji has suggested the interpolation method. We give below a somewhat straight proof.

To prove (8), we note that

$H\sin(x+1)° = (H\sin x°.H\cos 1°)/R \pm (H\sin 1°.H\cos x°)/R$

Now, put $H\sin 1° = 60'$, $H\cos 1° = 3437.4766$ and

$R = 3438'$ to get,

$$H\sin(x\pm1)° = (H\sin x°.3437.4766/3438) \pm (60H\cos x°/3438)$$

$$= H\sin x\ (1 - .5234/3438) \pm (10H\cos x/573)$$

$$= H\sin x(1 - 1/6568.5899) \pm (10/573)\ H\cos x$$

$$= H\sin x\ (1 - 1/6569) \pm (10\ H\cos x)/573$$

For the solution of diurnal problems the verse numbers (54), (55) and (57) of *Grhaganita* (*Spastādhikāra*) give the following formula :

(11) $H\sin\alpha = R(H\sin^2\lambda - H\sin^2\delta)^2 + H\cos\delta$

(12) $H\sin\alpha = H\sin\lambda \times H\cos\omega \div H\cos\delta$

Wherein λ, δ, α and ω are the longitude declination, right ascensian of the sun and obliquity of the ecliptic respectively. For details refer to Somayaji [1] and [2]

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 167-174

ON CONVERGENCE AND FIXED POINTS OF SEQUENCES OF MAPPINGS

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(Received 29.01,1992 and in revised form 02.12.1994)

ABSTRACT

Two results on common fixed points have been obtained. The first theorem deals with convergence of the sequence of common fixed points of four sequences of selfmappings on a metric space. In the second result an existence theorem for common fixed points of two sequences of mappings with two selfmappings has been established.

Mathematics subject classification (1991) : 54H25, 47H10

Key words and phrases : Weakly commuting mappings, sequences of mappings, common fixed point

INTRODUCTION

Several authors have dealt on the question of convergence of a sequence of fixed points or commom fixed points of mappings and many interesting results have been established. In the present paper we prove a theorem which gives sufficient conditions under which the uniform convergence of four sequences of mappings to four given mappings satisfying a Meir and Keeler type contractive condition implies the convergence of the common fixed points of the sequences of mappings to a common fixed point of the four mappings. The second theorem is an extension, to the case involving sequences of mappings, of the following recent theorem by the first author [8,p.78], [9]:

Theorem A Let P and S be weakly commuting mappings and Q and T be weakly commuting mappings of a complete metric space (X,d) into itself satisfying the conditions:

Given $\varepsilon > O$, there exists an $h(\varepsilon) > O$, $h(\varepsilon)$ being a nondecreasing function of ε such that for all x, y in X

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$\varepsilon \leq max\{d(Sx,Ty),\ d(Px,Sx),\ d(Qy,Ty),$

$[d(Px,Ty) + d(Qy,Sx)]/2\} < \varepsilon + h$

$\Rightarrow d(Px,Qy) < \varepsilon$ .....(i)

$Px = Qy$ whenever $Px = Sx$ and $Qy = Ty$. .....(ii)

Let the range of T contain the range of P and the range of S contain the range of Q. If one of the mappings P, Q, S and T is continuous then the mappings P,Q, S and T have a unique common fixed point which is also the unique common fixed point of P and S and of Q and T.

It is of interest to note that in the present extension of the above theorem we have assumed very few additional conditions.

Both of our results extend and unify several important fixed point theorems.

RESULTS

Two selfmappings F and G of a metric space (X,d) are called weakly commuting provided

$d(FGx,\ GFx) \leq d(Fx,Gx)$

for all x in X.

Theorem 1 Let P_n, Q_n, S_n and T_n be mappings of a metric space (X,d) into itself with at least one common fixed point u_n for each n = 1,2,... . Let P, Q, S and T be selfmsappings of X having a common fixed point u and satisfying the conditions:

Given $\varepsilon > O$, there exists $h(\varepsilon) > O$, $h(\varepsilon)$ being a nondecreasing function of ϵ, such that for all x,y in X

$\varepsilon \leq max\ \{d(Sx,Ty),\ d(Px,Sx),\ d(Qy,Ty).\ [d(Px,Ty) + d(Sx,Qy)]/2\} < \varepsilon + h$

$\Rightarrow d(Px,Qy) < \varepsilon$ .....(1)

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$Px = Qy$ whenever $Px = Sx$ and $Qy = Ty$. .....(2)

If the sequences of mappings $\{P_n\}$, $\{Q_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to P, Q, S and T respectively then the sequence $\{u_n\}$ converges to u uniquely.

Proof From (1), for all x, y in X for which $Px, \neq Sx$ and $Oy \neq Ty$, we have

$$d(Px,Qy) < max\ \{d(Sx,Ty),\ d(Px,Sx),\ d(Qy,Ty),$$

$$[d(Px,Ty) + d(Sx,Qy)]/2\} \qquad .....(3)$$

The point u_n is the common fixed point of P_n, Q_n, S_n and T_n fo reach $n = 1,2,\ldots$, so we have

$$P_n u_n = S_n u_n = u_n = Q_n u_n = T_n u_n.$$

Also, $Pu = Su = u = Qu = Tu$.

To establish the theorem, we first show that

$$lim_n\ d(Pu_n, Qu) = 0.$$

If $\lim_n d(Pu_n,Qu) \neq 0$, then there exist a subsequence $\{u_{n_i}\}$ of $\{u_n\}$, a positive integer N and a number $r > 0$ such that

$$n_i \geq N \Rightarrow inf\ \{d(P_{u_{n_i}}, Qu)\} = r.$$

Further, since the sequenes $\{P_n\}$, $\{Q_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to P, Q, S and T respectively, given $h = h(r) > 0$ and for each $u_n \in X$ there exists a positive integer $M = M(h(r))$ such that

$$n \geq M \Rightarrow d(P_n u_n,\ Pu_n) < h/4,\ d(Q_n u_n, Qu_n) < h/4.$$

$$d\ (S_n u_n,\ Su_n) < h/4,\ d(T_n u_n,\ Tu_n) < h/4 \qquad .....(4)$$

Therefore, for $n \geq N$ we obtain

$$d(Pu_n, Qu) < max\{d(Su_n, Tu),\ d(Pu_n, Su_n),\ d(Qu, Tu),\ [d(Pu_n, Tu) + d(Su_n, Qu)]/2\}$$

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$\leq d(S_n u_n, Su_n) + d(S_n u_n, Pu_n) + d(Pu_n, Tu)$

$= d(Pu_n, Qu) + d(S_n u_n, Su_n) + d(P_n u_n Pu_n)$

$< d(Pu_n, Qu) + h/2.$ .....(5)

Now, if we increase N then r and $h(r)$ do not increase, that is, if we increase N then M need not increase. Let us therefore, select $N > M$ and denote the corresponding value of r by r_N, that is,

$n_i \geq N \Rightarrow inf\{d(P_{u_n}, Qu)\} = r_N.$

This means there exists at least one value m of n_i such that $m \geq N$ and $d(Pu_m, Qu)$ is arbitrarily close to r_N and

$d(Pu_m, Qu) > r_N.$ .....(6)

This, in view of (5), implies

$r_N < max\{d(Su_m, Tu), d(Pu_m, Su_m), d(Qu, Tu), [d(Pu_m, Tu) + d(Su_m, Qu)]/2\}$

$< r_N + h(r_N).$ .....(7)

However, in view of (1) and (4), inequality (7) yields

$d(Pu_m, Qu) < r_N.$

This contradicts (6). Hence $lim_n\ d(Pu_n, Qu) = 0$. Then

$lim_n\ d(u_n, u) = lim_n\ d(P_n u_n, Qu)$

$\leq lim_n\ d(P_n u_n, P_n n) + lim_n\ d(Pu_n, Qu) = 0$

Therefore, the sequence $\{u_n\}$ converges to u. Proof of uniqueness of u follows easily.

Thus the theorem is established.

Remark 1 : On taking $S_n = T_n$ and $P_n = Q_n$ for every value of n, in the above theorem, we obtain a generalized version of Theorem 4 of Singh [14] as a special case of our theorem.

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Remark 2: In the above theorem, if we take $P_n = Q_n$ and $S_n = T_n$ = the identity mapping, we obtain Theorem 6.12 of Singh [15] as a particular case of our theorem. We also obtain theorem due to Dube and Singh (1972), see Singh [15], as a particular case of our theorem.

Remark 3 : In theorem 1, if we select $P_n = Q_n$ and $S_n = T_n$ = the identity map, then we obtain an extended form of the Theorem 4.6.8 of [4, p. 151] which is due to Bose and Mukherjee [1].

Theorem 2 : Let $\{P_i\}$ and $\{Q_j\}$, $i,j = 0,1,2, \ldots$, be sequences of selfmappings and let S and T be selfmappings of a complete metric space (X,d) satisfying the conditions:

Given $\varepsilon > 0$, there exists $h(\varepsilon) > 0$, $h(\varepsilon)$ being a nondecreasing function of ε, such that for all x, y in X

$$\epsilon \leq max\{d(Sx,Ty),\ d(P_ix,Sx),\ d(Q_jy,Ty),\ [d(P_ix,T_y) + d(Q_jy,Sx)]/2\} < \epsilon + h$$

$$\Rightarrow d(P_ix,\ Q_jy) < \epsilon \qquad \ldots\ldots(1)$$

$$P_ix = Q_jy \text{ whenever } P_ix = Sx \text{ and } Q_jy = Ty. \qquad \ldots\ldots(2)$$

Let the range of one of the mappings $\{P_i\}$, say Po, be contained in the range of T and the range of one of the mappings $\{Q_j\}$, say Q_0, be contained in the range of S. Also let P_0 and S, and Q_0 and T be weakly comuting. If one of the mappings P_0, Q_0, S and T be continuous then the mappings $\{P_i\}$, $\{Q_j\}$, S and T have a unique common fixed point which is also the unique common fixed point of $\{P_i\}$ and S and of $\{Q_j\}$ and T for every value of i and j.

Proof Let us first consider the mappings P_o, Q_o, S *and* T. These four mappings satisfy all the conditions of Theorem A. Therefore, by virtue of Theorem A, the mappings P_o, Q_o, S and T have a unique common fixed point, say z. Further, z is the unique common fixed point of P_o and S and of Q_o and T. Thus

$$P_oz = Q_oz = z = Sz = Tz.$$

Let us now consider a mapping P_i for some arbitrarily chosen value of i. Then $z = P_iz$. Otherwise, from (1) we get

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$d(P_i z, z) = d(P_i z, Q_o Z)$

$< max\{d(Sz,Tz), d(P_i z,Sz), d(Q_o z,Tz),$

$[d(P_i z,Tz) + d(Q_o z,Sz)]/2\}$

$= d(z,P_i z),$

a contradiction. Similarly it can be shown that $z = Q_j z$ for any value of j. If possible, suppose u is a second common fixed point of P_i and S. Then from (1) we obtain

$d(u,z) = d(P_i u.Q_0 z)$

$< max\{d(Su,Tz), d(P_i u,Su), d(Q_o z,Tz), [d(P_i u,Tz) + d(Q_o z,Su)]/2\}$

$= d(u,z),$

a contradiction. Hence z is the unique common fixed point of P_i and S for each value of i. Similarly z is the unique common fixed pont of Q_j and T for every value of j.

This establishes the theorem.

Remark 4 : In Theorem 2, if we let $P_i = Q_i$ for every value of i and $S = T =$ the identity mapping, we obtain a generalized version of a theorem due to Bose and Mukherjee [1].

Remark 5 : Letting $P_i = Q_i$ for each i and $S = T$ in Theorem 2, an extended form of Theorem 1 of Tewari and Singh [16] is obtained.

Remark 6 : By an obvious modification in the proof of Theorem 2, that is, by using the properties of compatibility in place of those of weak commutativity at relevant places, the theorem pertains to compatible maps and we obtain a multitude of fixed point theorems as special cases of our theorem. For example, if we take $P_i = P$ for each i, we can obtain the theorem due to Rhoades, Park and Moon [13] under relaxed conditions as a special case of our theorem. The theorem of Rhoades, Park and Moon assumes stronger condition on compatibility and the ranges of the mappings. The theorems due to Fisher [2], Hardy and Rogers [3], Jungck [5], Kannan [6], Meir and Keeler [7], Pant [8, p. 78], [9], [10], Park and Bae

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[11], Park and Rhoades [12], Singh [14], Tewari and Singh [16], [17] and Theorem 4.5.8 of [4] can also be obtained as special cases.

Remark 7 : A weaker form of Theorem 2, under the stronger condition that each P_i weakly commutes with S and each Q_j weakly comutes with T, can be obtained as a special case of Theorem 2.15 of [8, p. 89).

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 175-182

A FIXED POINT THEOREM FOR SELF MAPPINGS ON MENGER SPACES

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(Received 02.12.1994)

ABSTRACT

In this paper a fixed point theorem in the context of Menger Spaces is proved. The theorem is supported by an example. A fixed point theorem in complete metric spaces also follows.

AMS Subject Classification (1990) Primary 54H25.

Keywords : Menger Space, fixed point, t-norm.

INTRODUCTION

A Menger Space is defined as a triplet $(M,F,t,)$ where M is a set, $F:MxM\rightarrow \Delta^+$ (set of distribution functions) satisfies the following

(a) $F_{xy}(0) = 0$, for all $x,y \varepsilon M$
(b) $F_{xy}(s) = 1$, for all $s>0$ holds if and only if $x=y$
(c) $F_{xy} = F_{yx}$, for all $x,y \varepsilon M$
and (d) $F_{xy}(s_1 + s_2) \geq t(F_{xz}(s_1), F_{zy}(s_2))$, for all $x,y,z \varepsilon M$ and $s_1, s_2 \geq 0$

where t is a t-norm, that is a function $t:[0,1]x[0,1] \rightarrow [0,1]$ which satisfies
(i) $t(a,1) = a, t(0,0) = 0$
(ii) $t(a,b) = t(b,a)$
(iii) $t(c,d) \geq t(a,b)$, $c \geq a$, $d \geq b$
and(iv) $t(a,t(b,c)) = t(t(a,b),c)$ for all $a,b,c,d \varepsilon [0,1]$

Examples of t-norm are $t(a,b) = min(a,b), t(a,b) = max\{a+b-1,0\}$etc.

A Sequence $\{p_n\}$ is said to converge to p if given $\varepsilon>0$, $\lambda>0$ there exists a positive integer $N_{\varepsilon,\lambda}$ such that $F_{pp_n}(\varepsilon) > 1-\lambda$ whenever $n \geq N_{\varepsilon,\lambda}$.

A Sequence $\{p_n\}$ is a Cauchy sequence if given $\varepsilon>0$, $\lambda>0$ there exists

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a positive integer $N_{\varepsilon,\lambda}$ such that $F_{p_m p_n}(\varepsilon) > 1-\lambda$ whenever $m,n > N_{\varepsilon,\lambda}$.

A Menger Space is complete if every Cauchy sequence is convergent.

MAIN RESULTS

In this section we obtain a fixed point theorem for a self-mapping on a complete Menger space.

Lemma-1. $\{p_n\}$ is a sequence in a Menger Space $(M, F, t \equiv min)$ and $\{s_n\}$ is a sequence of positive real numbers such that $\sum_{i=0}^{\infty} s_i < \infty$ If $\prod_{i=0}^{\infty} F_{p_i p_i}(s_i) > 0$ then $\{P_n\}$ is a Cauchy sequence.

Proof. None of $F_{p_i p_{i+1}}(s_i) = 0$. So let, $F_{p_i p_{i+1}}(s_i) = 1-\lambda_i$ where $0 \le \lambda_i < 1$, i = 1, 2..... Then, $\prod_{i=1}^{\infty}(1-\lambda_i) > 0$ this implies ([3]) $\sum_{i=1}^{\infty} \lambda_i < \infty$. Given $\varepsilon > 0$ and $\lambda > 0$ we can find a positive integer $N_{\varepsilon,\lambda}$ such that

$$\sum_{i=N_{\varepsilon,\lambda}}^{\infty} s_i < \varepsilon \text{ and } \sum_{i=N_{\varepsilon,\lambda}}^{\infty} \lambda_i < \lambda$$

Let $m, n > N_{\varepsilon,\lambda}$. Further let $n > m$.

$$F_{p_m p_n}(\varepsilon) \ge F_{p_m p_n}(s_m + \ldots + s_{n-1})$$

$$\ge t\left(F_{p_m p_{m+1}}(s_m), F_{p_{m+1} p_n}(s_{m+1} + \ldots + s_{n-1})\right)$$

$$\ge t\left(F_{p_m p_{m+1}}(s_m), t\left(F_{p_{m+1} p_{m+2}}(s_{m+1}), t\left(\begin{matrix} F_{p_{m+2} p_{m+3}}(s_{m+2}) \ldots, \\ t\left(F_{p_{n-2} p_{n-1}}(s_{n-2})\, F_{p_{n-1} p_n}(s_{n-1})\right)\end{matrix}\right)\right)\right)$$

By virtue of properties of t-norm it follows that

$$F_{p_m p_n}(\varepsilon) \ge min\left\{F_{p_m p_{m+1}}(s_m), F_{p_{m+1} p_{m+2}}(s_{m+1}) \ldots, F_{p_{n-1} p_n}(s_{n-1})\right\}$$

$$\ge min\{1-\lambda_m, 1-\lambda_{m+1} \ldots, 1-\lambda_{n-1}\}$$

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or $F_{p_m p_{m+1}}(\varepsilon) > 1-\lambda$ whenever $m,n > N_{\varepsilon,\lambda}$. Thus $\{p_n\}$ is a Cauchy sequence.

Theorem-1. *Let (M,F,t ≡min)* be a complete menger space. T:M→M satisfies the following

a) $$F_{T_x T_y}(s_1 + s_2) \geq F_{xT_x}\left(\frac{s_1}{\alpha}\right) + F_{T_Y, T_y^2}\left(\frac{S_2}{\beta}\right) - 1$$

where $0 < \alpha < 1$ and $\beta > 0$ are constants and s_1 , $s_2 \geq 0$

and b) there exists $\{x_i\} \subset M$ and a sequence of real numbers $\{s_i\}$ with $\sum_{i=1}^{\infty} s_i < 1$ such that

$F_{x_i, T_{x_i}}(s_1) > 1 - s_1$

$F_{T_{x_2}, T^2 x_2}(s_2) > 1 - s_2$

and in general $F_{T_{x_i}^{i-1} T_{x_i}^{i}}(s_i) > 1 - s_i$, $i = 1,2...$

Then, T has a unique fixed point.

Proof. $F_{T_{x_i}^{i}, T_{x_{i+1}}^{i}}(s_i + \beta s_{i+1}) \geq F_{T_{x_i}^{i-1}, T_{x_i}^{i}}\left(\frac{s_i}{\alpha}\right) + F_{T_{x_{i+1}}^{i}, T_{x_{i+1}}^{i+1}}\left(\frac{\beta s_{i+1}}{\beta}\right) - 1$

Since, $0 < \alpha < 1, F_{T_{x_i}^{i}, T^{i}_{x_{i+1}}}(s_i + \beta s_{i+1}) \geq F_{T_{x_i}^{i-1}, T^{i}_{x_i}}(s_i) + F_{T_{x_{i+1}}^{i}, T^{i+1}_{x_{i+1}}}(s_{i+1}) - 1$

$> 1 - s_i + 1 - s_{i+1} - 1$

$= 1 - s_i - s_{i+1}$

Thus we have,

$F_{x_1, T_{x_1}}(s_1) > 1 - s_1$

$F_{T_{x_1} T_{x_2}}(s_1 + \beta s_2) > 1 - (s_1 + s_2)$

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$F_{T_{x_2}T^2_{x_1}}(s_2) > 1 - s_2$,

$F_{T^2_{x_2}T^2_{x_3}}(s_2 + \beta s_3) > 1 - (s_2 + s_3)$,

$F_{T^2_{x_3}T^3_{x_3}}(s_3) > 1 - s_3$

$F_{T^3_{x_3}T^3_{x_4}}(s_3 + \beta s_4) > 1 - (s_3 + s_4$ .....etc.

Also $0 < s_1, (s_1+s_2), s_2, (s_2+s_3)$.....$< 1$ because $\sum_{i=1}^{\infty} s_i <$

Now the series, $s_1 + (s_1+s_2) + s_2 + (s_2+s_3) + (s_3+s_4) +$....to
$= 2s_1+3s_2+3s_3+$.....to is convergent

So the infinite product,

$(1-s_1)(1-s_1-s_2)(1-s_2)(1-s_2-s_3)$.....$to\ \infty > 0$ ([3])

Hence, $F_{x_1.T_{x_1}}(s_1) F_{T_{x_1},T_{x_2}}(s_1 + \beta s_2).F_{T_{x_2},T^2_{x_2}}(s_2)$

$.F_{T^2_{x_2},T^2_{x_3}}(s_2 + \beta s_3)$....$to\ \infty > 0$

Also the series,

$s_1 + (s_1 + \beta s_2) + s_2 + (s_2 + \beta s_3) + s_3 + (s_3 + \beta s_4) + s_4 +$.... $to\ \infty$.

$= 2s_1 + (2 + \beta)\ s_2 + (2 + \beta)\ s_3 +$ .....to ∞

is convergent, since $\sum_{i=1}^{\infty} s_i <$

Therefore by Lemma-1 the sequence

$\{ x_1, T_{x_1}, T_{x_2}, T^2_{x_2}, T^2_{x_3}, T^3_{x_3}, T^3_{x_4}, T^4_{x_4} \ldots T^{i-1}_{x_i}, T^i_{x_i}, \ldots \}$

is convergent. Let the sequence converge to x.

Let $\varepsilon > 0$ be arbitrary.

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We choose α_1 such that $\alpha < \alpha_1 < 1$,

then $F_{xT_x}(\varepsilon) \geq t(F_{xT_{x_i}^{i-1}}((1-\alpha_1)\varepsilon)), F_{T_{x_i}^{i-1}T_x}(\alpha_1\varepsilon))$

$= t(F_{xT_{x_i}^{i-1}}((1-\alpha_1)\varepsilon), F_{T_{x_i}^{i-1}T_x}(\alpha_2 + (\alpha_1 - \alpha_2)\varepsilon)$

(where we choose α_2 such that $\alpha < \alpha_2 < \alpha_2$)

$$\geq t(F_{x,T_{x_i}^{i-1}}((1-\alpha_1)\varepsilon), F_{x,T_x}(\frac{\alpha_2}{\alpha}\varepsilon) + F_{T_{x_i}^{i-1},T_{x_i}^{i}}(\frac{\alpha_1-\alpha_2}{\beta}\varepsilon)) - 1)$$

Let n be an arbitrary positive integer. Since the sequence

$\{x_1, T_{x_1}, T_{x_2}, T_{x_2}^2, T_{x_3}^2, T_{x_3}^3 \ldots\}$

converges to the point x we can choose i in such a way that

$$F_{T_{x_i}^{i-1},T_{x_i}^{i}}(\frac{\alpha_1-\alpha_2}{\beta}\varepsilon) > 1 - \frac{1}{n}$$

and $F_{xT_{x_i}^{i-1}}((1-\alpha_1)\varepsilon) > 1 - \frac{1}{n}$,

Therefore $F_{xT_x}(\varepsilon) \geq t(1-\frac{1}{n}, F_{xT_x}(\frac{a_2}{\alpha}\varepsilon) + 1 - \frac{1}{n} - 1)$

$$\geq t(1-\frac{1}{n}, F_{xT_x}(\frac{a_2}{\alpha}\varepsilon) - \frac{1}{n}$$

$$\geq t(F_{xT_x}(\frac{a_2}{\alpha}\varepsilon) - \frac{1}{n}, F_{xT_x}(\frac{a_2}{\alpha}\varepsilon) - \frac{1}{n})$$

$$F_{xT_x}(\varepsilon) \geq F_{xT_x}(\frac{a_2}{\alpha}\varepsilon) - \frac{1}{n}$$

Since n is arbitrary

$$F_{xT_x}(\varepsilon) \geq F_{xT_x}(\frac{a_2}{\alpha}\varepsilon)$$

Since $\frac{a_2}{\alpha} >$ we have $F_{xT_x}(\varepsilon) = 1$ for all $\varepsilon > 0$.

That is $x = T_x$ is a fixed point. If possible, let $y = T_y$ be another fixed point of T.

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Then from a) for $s > 0$

$$F_{T_x T_y}(s) \geq F_{x T_x}\left(\frac{s}{2\alpha}\right) + F_{T_y T_y^2}\left(\frac{s}{2\beta}\right) - 1$$

$$or \quad F_{x,y}(s) \geq F_{xx}\left(\frac{s}{2\alpha}\right) + F_{y,y}\left(\frac{s}{2\beta}\right) - 1$$

$$or \; F_{x,y}(s) \geq 1 \; for \; s > 0.$$

Hence $x = y$
Therefore the fixed point is unique.

Cor-1. If (X,d) is a complete metric space and $T : \to X$ satisfies
1. $d(T_x T_y) < \alpha d(x, T_x) + \beta d(T_y, T^2_y)$, where $0 < \alpha < 1$ and $\beta > 0$
11. T is onto
111. inf $d(x, T_x) = 0$
Then T has unique fixed point

The proof of the Corollary will follow from the fact that a complete metric space can also be treated as a complete menger space if we put $F_{pq}(t) = H(t\text{-}d(p, q))$ [4], [6]. The inequality (1) in corollary implies the inequality (a) in the Theorem and conditions (11) and (111) in the corollary imply condition (b) of the Theorem. The proof then follows by the application of Theorem 1.

It may noted that fixed points on Menger space have been discussed in many works. We have noted some of them in references [1], [2] [5], [6].

Examples

We set an example to show the applicability of Theorem-1.
Let G any distribution function with $G(0)$ abd $0 < G(x) < 1$ for all $x > 0$. We consider the triplet (M,F,t) where $M = [0, \infty]$ t = min and F is defined as

$$F_{pq} = H, \text{ if } p = q$$

where $H(x) = 1, \; x \; . \; 0$
$\quad = 0, \; x < 0$
and $F_{pq}(s) = G(s/\ |p - q|)$
It is easily seen that (M,F,t) is a complete Menger space.

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Let $T : X \to X$ be defined as $T_x = x/4$. Then

$$|T_x - T_y| = \left|\frac{x}{4} - \frac{y}{4}\right| = \frac{1}{4}|x - y| \le \frac{1}{4}(|x| + |y|)$$

or, $|T_x - T_y| \le \frac{1}{4}.\ 2\ max\{|x|\ ,\ |y|\}.$

Therefore either Case I : $|T_x - T_y| \le \frac{1}{2}|x|$

or Case II : $|T_x - T_y| \le \frac{1}{2}|y|$

Case I : $|T_x - T_y| \le \frac{1}{2}|x| = \frac{1}{2}\frac{4}{3}\left|\frac{3}{4}x\right|$

$$\frac{2}{3}\left|x - \frac{1}{4}x\right| = \frac{2}{3}|x - T_x|$$

$$G\left(\frac{s}{|T_x - T_y|}\right) \ge G\left(\frac{s}{\frac{2}{3}|x - T_x|}\right),\ s > 0$$

$${}^{F}T_x, T_y(s_1 + s_2) \ge F_x, T_x\left(\frac{s_1}{2/3}\right) + {}^{F}T_y, T_y^2\left(\frac{s_2}{8/3}\right)$$

Case II : $|T_x T_y| \le \frac{1}{2}|y| = \frac{1}{2}\frac{16}{3}\left|\frac{3}{16}y\right|$

$$= \frac{8}{3}|y/4 - y/16| = \frac{8}{3}|T_y - T_y^2|$$

so $G(\frac{s}{|T_x - T_y|}) \ge G(\frac{s}{8/3|T_y - T_y^2|})s > 0$

or $F_{T_x T_y}(s) \ge F_{T_y, T_y^2}(\frac{s}{8/3})$

Both cases imply inequality

$$F_{T_x, T_y}(s_1 + s_2) \ge F_x, T_x\left(\frac{s_1}{2/3}\right) + F_{T_y}, {}_{T_y^2}\left(\frac{s_2}{8/3}\right)$$

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$$\geq F_{x,T_x}\left(\frac{s_1}{2/3}\right) + F_{T_y,T_y^2}\left(\frac{s_2}{8/3}\right) - 1$$

where $s_1, s_2 > 0$.

Let $\{s_i\}$ be a sequence such that $\sum_{i=1}^{\infty} s_i <$. we choose $\{x_i\}$ such that

$$G(s_i / \left|x_i\left(\frac{1}{4^i} - \frac{1}{4^{i-1}}\right)\right|) > 1 - s_i,\ i = 1,2$$

so that $F_{T_{x_i}^{i-1}, T_{x_i}^{i}}(s_i) > 1 - s_i,\ i = 1,2...$

Hence b) of Theorem-1 is satisfied with the choice of sequence $\{x_i\}$ and $\{s_i\}$

Thus by application of Theorem-I it follows that T has a unique fixed point. It may be seen that `$O$` is the unique fixed point.

ACKNOWLEDGEMENT

I am grateful to Prof. J.N. Das, Deptt. of Applied Mathematics, Calcutta University, for his guidance and advice during the preparation of the paper.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 183-186

FIXED POINTS IN FUZZY METRIC SPACES

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(Received 21-12-1994)

ABSTRACT

In this note we establish Banach type fixed point theorems on fuzzy metric spaces (FM-spaces). Our results generalize several known fixed point theorems on FM-spaces and extend certain results of Hicks-Rhoades and others.

Mathematics Subject Classifications (1991) : 54H25, 54A40, 54E99.

Keyword and Phrases : Fuzzy metric space, f ixed point, orbitally upper semicontinuous.

INTRODUCTION

Following Kramosil-Micha'lek's [5] approach to fuzzy metric spaces (FM-spaces), Grabiec [2] has formulated fixed point theorems for contractive maps on FM-spaces. Subsequently, Fang [1], Mishra et al. [6] and Singh et al. [7] have investigated existence of solutions of abstract fixed point equations on FM-spaces. The intent of this note is to establish fixed point theorems for Banach type maps on FM-spaces. Our results extend and fuzzify fixed point theorems of Hicks-Rhoades [3], Hicks [4] and others.

PRELIMINARIES

We shall generally follow the notations and definitions of Grabiec [2], Hicks-Rhoades [3], and Mishra et al. [6].

A fuzzy metric space is an ordered triplet $(X, M, *)$ consisting of a nonempety set X, a fuzzy set M in $X^2 \times (0, \infty)$ and $*$, a continuous t-norm [2] (see, also [5], [6]). The functions $M(x, y, .) : [0, \infty) \to [o, 1]$ are left continuous and are assumed to satisfy the following conditions :

(*FM-1*) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y$

(*FM-2*) $M(x, y, 0) = 0$

(*FM-3*) $M(x, y, t) = M(y, x, t)$

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(*FM-4*) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$, *for all* $x, y, z \in X$ and $t, s > 0$.

Grabiec [2] has shown that $M(x, Y, *)$ is nondecreasing for all $x, y \in X$. For details of topological preliminaries, refer to [2] and [5].

A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim M(x_n, x, t) = 1$ for each $t > 0$, and $\{x_n\}$ is a Cauchy sequence if $\lim M(x_{n+p}, x_n, t) = 1$ for each $t > 0$, and $p > 0$. An FM-space, in which every Cauchy sequence is convergent, is called complete.

In all that follows, X stands for an FM-space $(X, M, *)$ with the following property :

(*FM-5*) $\lim_{t \to \infty} M(x, y, t) = 1$ *for all* $x, y \in X$.

FIXED POINT THEOREMS

Defination 1. Let X be an FM-Space and $T : X \to X$. Orbit of T at x, denoted by $O(x, \infty)$ is defined as $O(x, \infty) = \{x, Tx, T^2x.....\}$.

Defination 2. Let X be an FM-space. A function G from X to $[0, 1]$ is *T-orbitally upper semicontinuous at* $z \in X$ if $\{x_n\}$ is a sequence in $O(x, \infty)$ and $\{x_n\}$ converges to z implies $G(z) \geq \limsup G(x_n)$.

The following result is essentially due to Grabiec [2].

Theorem 1. Let X be an FM- space. If $T: X \to X$ satisfies
(1.1) $M(Tx, Ty, kt) \geq M(x, y, t)$;
for all $x, y \in X$, *and* $0 < k < 1$, *then* T *has a unique fixed point.*

The above result is known as *fuzzy Banach contraction principle.*

Theorem 2. *Let* X *be an FM-space and* $T : X \to X$. *If* X *is T-orbitally complete and for an* $x \in X$,
(2 .1) $M(Ty, T^2 y, kt) \geq M(y, Ty, t)$;
$y \in O(x, \infty)$, $k \in (0, 1)$, *then for an* $x_o \in X$, *there exists an orbit* $O(x_o)$ *converging to a point* $z \in X$. *Further :*

(*a*) z *is a fixed point of* T *if* T *is orbitally continuous.*

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(b) z is a fixed point of T if and only if $G(x) = M(x, Tx, t)$ is T-orbitally upper semicontinuous at z.

Proof. Pick $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that $x_1 = Tx_0, x_2 = Tx_1, \ldots, x_n = Tx_{n-1} = T^n x_0$
In view of (2.1),

$$M(x_n, x_{n+1}, t) = M(Tx_{n-1}, T^2x_{n-1}, t) \geq M(x_{n-1}, Tx_{n-1}, t/k) \geq \ldots \geq M(x_0, x_1, t/k^n).$$

This shows that $\{x_n\}$ is a cauchy sequence in X, hence convergent. Call the limit z.

(a) Since
$M(Tz, z, t) \geq M(Tz, Tx_n, t/2) * M(x_{n+1}, z, t/2)$

and T is continuous, making $n \to \infty$ yields $M(Tz, z, t) \geq 1*1 = 1$

Thus z is a fixed point of T.

(b) Let $z = Tz$ i .e $M(z, Tz, t) = 1$ Then $G(z) = M(z, Tz, t) = 1 \geq \lim \sup G(x_n)$, implying $G(x)$ is T-orbitally upper upper semicontinuous at z.

Conversely, if $G(x)$ is T-orbitally upper semicontinuous at z, then $G(x_n) = M(x_n, Tx_n, t) \leq M(z, Tz, t)$, yielding
$M(z, Tz, t) = 1$, *i.e.*, $z = Tz$.

Following theorem shows that Theorem 2 is more general than Theorem 1.

Theorem 3. *Let X be an FM-space and $T: X \to X$. Then the condition (1.1) implies the condition (2.1)*

Proof. Taking $y = Tx$ in (1.1), we get the condition (2.1).

Here we remark that our result is superior to several results on the existence of fixed points in FM-spaces. But, first, consider the following for $T: X \to X$. For all $x, y \in X$,

(3.1) $M(Tx, Ty, kt) \geq M(x, y, t) * M(x, Tx, t) * M(y, Ty, t) * M(x, Ty, \alpha t) * M(y, Tx, (2-\alpha)t)$.

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where $k \in (0, 1)$, $t > 0$ and $\alpha \in (0, 2)$.

Recently Mishra et al. [6] and Singh et al. [7] studied the condition (3.1) and obtained some interesting results on FM-spaces. Following comparing theorem shows that our result also generalize several fixed point theorems for above contractive condition.

Theorem 4. Let X be an FM-space and $T: X \to X$. Then the condition (3.1) implies the condition (2.1).

Proof. Taking $y=Tx$ in (3.1). We get for $\alpha = 1+q$, $q \in (0, 1)$

$$M(Tx, T^2x, kt) \geq M(x, Tx, t) * M(Tx, T^2x, t) * M(x, T^2x, (1+q)t) * M(Tx, Tx, (1-q)t)$$
$$\geq M(x, Tx, t) * M(x, Tx, qt).$$

Since the norm * is continuous and $M(x, y, .)$ is left continuous, making $q \to 1$ gives $M(Tx, T^2x, kt) \geq M(x, Tx, t)$. This completes the proof.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 187-190

ON SOME PROPERTIES OF DISTANCE SETS

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(Received 21-12-94)

ABSTRACT

Steinhaus proved that the set of differences A-B contains an interval whenever A and B are linear sets of positive Lebesgue measure. S.Piccard proved that the same conclusion holds when A and B have the property of Baire. In the present note we strengthen the results of Steinhaus as well as Piccard.

Mathematics subject classification (1991): 28A 05

Key words and Phrases : Distance set, Property of Baire, Residual set.

INTRODUCTION

Let A and B be subsets of the real line R. Then the distance set of A and B is defined to be the set $D(A,B) = \{|x-y|: x \in A, y \in B\}$. The investigation of the set of distances between the points of a measurable set was started by H. Steinhaus. In 1920, Steinhaus [7] proved that if A, $B \subset R$ be sets with positive Lebesgue measure then $D(A,B)$ contains an interval. A linear set is said to have the "Property of Baire" if it can be expressed as the symmetric difference of an open set and a set of first category. A set of the second category which is complement to a set of the first category is called a residual set [4]. Therefore, there are two kinds of sets of the second category, those which are complement to a set of the first category and those for which this is not the case. It is also known that the intersection of a sequence of residual sets is also residual. In 1939, S. Piccard [5] proved the result same as that of Steinhaus when A and B were two linear sets of second category having the property of Baire. Many papers are devoted to the study of the distance set $D(A)$ for various A. The corresponding problem for sets in n-dimensional space was considered by Kestelman [3] in 1947, and he proved a number of elegant theorems about distance sets in n-dimensional space and Steinhaus theorem came out as a particular case of one of his results. The question has been asked if there is

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a set of measure zero for which the distance set contains an interval with origin as a left hand end point. In 1917, Steinhaus [6] proved that the Contor set C is an example of a set of measure zero whose distance set contains an interval with origin as a left hand end point. Bose Majumdar [2] made an exhausted study of $D(C)$. Boas [1] has proved that for almost all d in $[0,1]$ there is an uncountable (of the power of the continuum) set of pairs $(x,y) \in C \times C$ such that $|x-y| = d$. In connection with this, the question arises what is the situation for the sets of positive measure.

Steinhaus [7] proved the following interesting result: *Let A and B be linear sets with positive measure and C be a set of real numbers everywhere dense; then there exist two points 'a' and 'b' belonging to A and B respectively such that the number $|a-b|$ belongs to C.* As a consequence of this result he proved that $D(A,B)$ for sets of positive measure A and B, contains an interval.

In this paper we have attempted to generalise this result. For some d in a countable dense subset of R, the equality $|x-y| = d$ holds for an uncountable (of the power of the continuum) number of pairs $(x,y) \in A \times B$. We obtain the same result as above when A and B are taken as sets of second category having the property of Baire.

Theorem 1. *Let A and B be two linear sets of positive measure and C be a countably dense subset of R. Then for some $d \in C$, the set $E = \{x \in A: \exists\, y \in B$ with $|x-y| = d\}$ has positive measure.*

Proof. If possible let E be of measure zero. Then for each $d \in C$, the set $\{x \in A: \exists y \in B$ such that $|x-y| = d\}$ has measure zero. Therefore the set $F = \{x \in A: \exists y \in B$ such that $|x-y| = d,\ d \in C\}$ also has measure zero.

Let $H = A \setminus F$. Then $\lambda(H) = \lambda(A) > 0$, where λ denotes the Lebesgue measure. Then according to Steinhaus [7] the set $B \setminus H$ contains an interval. Hence $C \cap (B \setminus H) \neq \Phi$. Then for some $y \in B$, $c \in C$ *and* $x \in H \subseteq A$ we have $y = x + c$. Thus $x \in F - a$ contradiction. Hence the result.

Note. For some d in a countably dense subset C of R, the equality $|x-y| = d$ holds for an uncountable (of the power of continuum) number of pairs $(x,y) \in A \times B$.

Steinhaus [7] proved that $D(A,B)$ contains an interval where A and B are two linear sets of positive measure. Let us define the set $D_1(A,B)$ consisting

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of all those points d which can be expressed into the form $d=|x-y|$ for infinite number of pairs of points $x\in A$, $y\in B$. Obviously, $D_1(A.B)\subset D(A,B)$.

We have established the following result which is an easy consequence of Steinhaus's result :-

Theorem 2. If A and B are two linear sets of positive measure, then $D_1 \equiv D_1(A,B)$ contains at least one interval.

Proof. Suppose not, then D_1 does not contain an interval. So $R\backslash D_1$ is a dense subset of R. Then, we may choose C to be countably dense subset of $R\backslash D_1$. By the note after theorem 1 there exists $d\in C$ such that d is the distance between infinite number of points of A and B. Hence $d\in D_1$--*a* contradiction.

Therefore, D_1 contains at least one whole interval.

Theorem 3. *Let A and B be two linear sets of second category having the property of Baire. If C is a countable dense subset of R, then for some $d\in C$, the set $\{x\in A: x\pm d\in B\}$ is of second category.*

Proof. If not, then for every $d\in C$, the set $F_d=\{x\in A: x\pm d\in B\}$ is of first category.

Then the set $\underset{d\in C}{U} F_d$ is also of first category in R.

Therfore $E = A - \underset{d\in C}{U} F_d$ is a residual set in R.

Then according to Piccard [5], $B\backslash E$ contains an interval. Hence, $C\cap(B\backslash E)\neq\Phi$. Then for some $d\in C$, $b\in B$ and $a\in E\subseteq A$ we have $b=a+d$ and hence $a\in F_d$--*a* contradiction.

Hence the result.

Theorem 4. *Let C be a countable dense subset of R and I be a set of first category. Then there exists a residual set P such that, $\forall x\in P$, $x+c\notin I$ for any $c\in C$.*

Proof. Let us enumerate C as $e_1, e_2, \ldots\ldots e_n, \ldots\ldots$ Let $R_n=\{r\in R: e_n+r\notin I\}$ $(n=1,2,\ldots.)$. Then $R_n=[R-I]\ (-e_n)$, which is a residual set for every n. *Let*

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$P = \bigcap_{n=1}^{\infty} R_n$ Then P is a residual set and $\forall x \in P$, $x+e_n \notin I$ for any n.

Hence the proof.

Theorem 5. *If A and B are two residual linear sets in some open interval, then the set of all points d for which* $\{x \in A : x+d \in B\}$ *is of second category, contains an interval.*

Proof. Let E be the set of all those points d for which the set $\{x \in A : x+d \in B\}$ is of second category.

We first show that E hits every countable dense subset C of R.

If not, then for each $d \in C$, the set $\{x \in A : x+d \in B\}$ is of first category. Then set $F=\{x \in A : x+d \in B,\ d \in C\}$ is also of first category. Then $G=R \setminus F$ is a residual set having the property of Baire. So according to Piccard [5], $B \setminus G$ contains an interval. Then $(B \setminus G) \cap C \neq \Phi$, which gives rise to a contradiction.

Now, if E contains no interval, then there exists a countable dense subset C of R such that $E \cap C = \Phi$-- which is again a contradiction.

Hence the theorem.

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 191-206

NONLINEAR HYBRID CONTRACTIONS

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(Received 30-03-1993)

ABSTRACT

Standard fixed point theorems state conditions under which there exist solutions to $Tx = x$ for the single-valued self-map T of a space X or to $y \in Py$ for a multivalued map P on X with values in its power set. In this paper, inter alia, we present a brief historical account of coincidence/fixed point theorems for contracting single- and multi-valued maps.

Mathematics Subject Classifications (1991) : 54H25, 47HIO.

Keywords and Phrases Coincidence/fixed point, multivalued map, weakly commuting maps, compatible maps.

INTRODUCTION

The well-known Brouwer fixed point theorem (*viz. a continuous map on a closed unit ball of R^n has a fixed point*) has maintained its magnificient presence in mathematics for the last 80 years, excluding, of course, a gestation of 2-3 years. This "fixed point"magnet may be called the queen of fixed point theory and, perhaps, of macro-analysis and its applications. It has many beautiful children, grand and grand-grand children such as, with the risk of missing many champion theorems, Birkhoff-Kellog (1922). Lefschetz (1926). Schauder (1927). Tychonoff (1935). Kakutani (1941), Hukuhara (1950), Glicksberg (1952), Darbo (1955), Browder (1959, 67), Jones (1963), Sadovskii (1967), Ky Fan (1961, 66, 69, 84). Reih (1972, 78, 79), Lin (1979), V.M. Sehgal & S.P. Singh (1985) and Bridges et al. (1992, cf. [5]). One may refer [41]) and [52] for the recent development of these theorems. However, beauty (of these results) has to pay the price of

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* A major part of this article presented at the International Conference on Modern Analysis and Applications, I.I.T., New Delhi (11-14 December 1992).

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cosmetics in kind of "compactness in some sense." " - - - the spaces which have the fixed point property for continuous copact set-valued mappings constitute a fairly small subclass of those which has the fixed point proerty for continuous single-valued mappings; (see Nadler [33, p. 476])." This, perhaps, inspires the development of fixed point theory for contracting maps. In this paper, we are concerned about the study of fixed points of contracting multivalued maps "majorized" by single-valued maps, which started about a decade ago. A *prima facie* love for undertaking such a study appears to be H.W. Corley's formulation [7, Th. 1] of an abstract maximization problem in terms of a (hybrid) stationary point theorem.

PRELIMINARIES

We will use the following notation where (X, d) is a metric space :

$CL(X) = \{A \subset X: A \text{ is nonempty and closed}\}$,

$CB(X) = \{A \subset X: A \text{ is nonempty, closed and bounded}\}$,

For nonempty subsets A, B of X and $s > 0$,

$N(s, A) = \{x \in X : d(x, a) < s \text{ for some } a \in A\}$,

$E_{A,B} = \{s>0 : A \subset N(s,B), B \subset N(s,A))\}$,

$$H(A,B) = \begin{cases} inf\, E_{A,B} & \text{if } E_{A,B} \neq \phi \\ +\infty & \text{if } E_{A,B} = \phi, \end{cases}$$

and for any $x \in A$, $d(x, A)$ denotes the distance between x and A. H is called the *Hausdorff* (resp. *generalized Hausdorff*) distance function for $CB(X)$ (resp. $CL(X)$) induced by d. In all that follows *id* stands for the identity map on X.

Definition 1 [34], [48]

Maps $f: X \to X$ and $T: X \to CL(X)$ are weakly commuting at $z \in X$ if $H(fTz, Tfz) \leq d(fz, Tz)$. f amd T commute weakly on X if they commute weakly at every point of X.

For an equivalent formulation for two single-valued maps on a metric space, see [44].

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Definition 2 [24]

Maps $f : X \to X$ and $T : X \to CB(X)$ are compatible iff $fTx \in CB(X)$ for all $x \in X$ and $H(Tfx_n, fTx_n) \to 0$ whenever x_n is a sequence in X such that $Tx_n \to M \in CB(X)$ and $fx_n \to t \in M$.

For an equivalent formulation for two single-valued maps on a metric space refer [19-20]. We remark that commuting maps are weakly commuting and weakly commuting maps are compatible, and the reverse implication is not true (see, for instance, [19-20], [24], [34], [44-45] and [47-48]). However, all the three concepts for a pair of maps are equivalent at a concidence point of the maps.

Let $T : X \to CL(X)$ and $f : X \to X$. We define (T, f) to be:

hybrid contraction if

(1) $H(Tx, Ty) \leq kd(fx, fy)$ for some $0<k<1$, and x,y in X;

hybrid nonexpansive if

(2) $H(Tx, Ty) \leq d(fx, fy)$, x,y in X; and

hybrid contractive if

(3) $H(Tx, Ty) < d(fx,fy)$, $fx \neq fy$ and x,y in X.

These definitions are modeled respectively on Banach contraction, nonexpansive and Edelstein's contractive maps [9-10] when $T : X \to X$ and $f = id$. Maps satisfying variants of (1)-(3) will be called hybrid type contraction/nonexpansive/contractive. For example, T and f satisfing.

(4) $H(Tx, Ty) \leq k \max \{d(fx,fy), d(fx, Tx), d(fy, Ty), (1/2)[d(fx, Ty)+d(fy, Tx)]\}, 0<k<1$ and $x,y \in X$,

will be called **'hybrid type contraction'.** If no distinction is needed in a particular situation, maps falling in either of the three or similar categories will generally be described as **hybrid contracting maps** (Note that (1) implies (i), $i = 2,3,4$.)

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The study of hybrid contracting maps was initiated during 1981-83 by Bhaskaran-Subrahmanyam [3] (see also [43], p.3), Hadzic [12], Kaneko [21], Mukherjee [32], Naimpally et al. [35] and Singh-Kulshrestha [49]. Indeed, the following coincidene theorem has a constructive proof in [49].

Theorem 1

Let $T: X \to CL(X)$ and $f: X \to X$ be such that (4) obtains, $f(X)$ is a complete subspace of X and

(1.1) $T(X) \subset f(X)$.

Then T and f have a coincidence. Indeed, for any x_o in X, there exists a sequence $\{x_n\}$ in X such that

(i) $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$;

(ii) $\{fx_n\}$ converges to fz for some z in X, and $fz \in Tz$,

i.e., T and f have a coincidence at z; and

(iii) $d(fx_n, fz) \leq (k^{1-q})^n (1-k^{1-q})^{-1} d(fx_o, fx_1)$,

where $q < 1$ is any positive number.

Coincidence theorems for (T,f) satisfying (1) and (1.1) appear in [12], [21] and [32] under strong conditions (such as continuity, commutativity) on the maps, Extensions of Theorem 1 appear in [42-43] and [48]. We remark that the condition (1.1) is sufficient for the existence of a sequene $\{fx_n\}$ but not necessary. Theorem 1 with $f=id$ is an important generalization of the multivalued contraction principle [8, 23] due to 'Ciric' [6, Th.2]. Indeed, the condition (4) of Theorem 1 includes several contraction type conditions for a multivalued map T on X with values either in $CB(X)$ or $CL(X)$ studied among others by Covitz-Nadler [8], Iseki [15], Kaneko [22-23], Kita [26, p. 115]. Nadler [33], Ray [39], Reich [40] and Wegrzyk [57]. Further, using 'Ciric's result [op.cit.] and following [35, Th. 2], one may also prove that T and f under the conditions of Theorem 1 have a coincidence.

Example 1

Under the conditions of Theorem 1, one can not expect, in general, a

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common fixed point for the maps T and f. In fact, let $X = [0, \infty)$, $Tx = [1+x, \infty)$ and $fx = 2x$ for x in X (see [35], [43]). Then all the hypotheses of Theorem 1 are satisfied, and T, f are common fixed point free. Indeed, T has no fixed point in X.

Example 2

Condition (1.1), although sufficient, is not necessary to obtain a coincidence. Let $X = [0, \infty)$, $Tx = [0, e^x]$ and $fx = 2(e^x-1)$, then $f(X)$ is complete, (4) obtains, (1.1) fails and T and f have a coincidence, in fact, a common fixed point.

SOME HISTORICAL NOTES

The condition (1) with T single-valued on X was first studied by R. Machuca [29] in 1967 followed by Goebel [11] in 1968 (see (1*) below). In a symposium on Fixed Point Theory and its Applications (Meerut University, Meerut, 1982), papers of Bhaskaran-Subrahmanyam and Singh-Kulshrestha attempting to investigate conditions under which (T, f) satisfying (1)/(4) could have a common fixed point were a subject of discussion (see also [43], p. 3). Mukherjee [32] attempted, in vain, to establish a common fixed point theorem for continuous and commuting maps T and f satisfying (1) and (1.1) with X complete. Olga Hadzic [12] studied the following hybrid type contraction condition for $f, g : X \to X$ and $T : X \to CB(f(X) \cap g(X))$:

(5) $H(Tx, Ty) \leq kd(fx, gy)$, $x, y \in X$,

and attempted a solution to $fx \in Tx$ and $gx \in Tx$, wherein f, g are continuous and T is closed. Hadzic's result is a special case of Theorem 4 (see below).

Let Y be an arbitrary set and $f, g: Y \to X$ such that for $0<k<1$ and x, y in Y,

(1*) $g(Y) \subset f(Y)$ and $d(gx, gy) \leq kd(fx, fy)$.

Machuca [op. cit.] attempted to show the existence of a coincidence point of f and g satisfying (1*) under heavy topological conditions. Goebel

+ In a chat (Thunder Bay, 1983), Prof. G. Jungck denied the knowledge of [29]and [11].

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[op. cit.] using the Banach contraction principle proved that the maps f and g have a coincidence when (1*) obtains and $f(Y)$ is complete. In 1976, Jungck [17], perhaps unaware+ of Machuca [op. cit.] and Goebel [op. cit.], gave a constructive proof to show that continuous commuting maps f and g satisfying (1*) with $Y=X$ complete have a unique common fixed point. This fixed point theorem together with [18] led to a massive growth of fixed point theorems for commuting, weakly commuting and compatible maps; we cite [2-4], [12-13], [19-21], [24], [28], [31-32], [34-35], [37-38] and [42-50] as examples.

A metric space X is metrically convex if for any distinct x, y in X, there exists z in X distinct from x,y such that $d(x,z) + d(z,y) = d(x,y)$. Let X be a complete and metrically convex metric space, and K a nonempty closed subset of X. Let $f: K \to X$ be a continuous map and $S, T: K \to CB(X)$ such that for all x,y in K,

$$\text{(BS)}\quad H(Sx, Ty) \le \alpha\, d(fx, fy) + \beta\,[d(fx,Sx) + d(fy, Ty)] + \gamma\,[d(fx, Ty) + d(fy, Sx)]$$

Where $\alpha, \beta, \gamma \ge 0$ with $(\alpha + \beta + \gamma)(1 + \alpha + \gamma)/(1-\beta-\gamma)^2 < 1$. Assad-Kirk [1] studied (BS) with $S = T, f = id$ and $\beta = \gamma = 0$ by levying certain boundary conditions on the map T. Subsequently, Itoh [16], Khan [25], Tan-Minh [55], Yanagi [58] and others generalized their result [1, Th.1] for multivalued maps on K with values in $CB(X)$. Initial coincidence/fixed point theorems under (BS) and (BS) with $S = T$ obtained in [4] extend certain nice fixed point theorems of Itoh [op.cit.] and Khan [op. cit.].

Following Assad-Kirk [1], Bhaskaran-Subrahamanyam [4] (see also [3]) initiated the study of hybrid nonexpansive maps T and f on a subset of a banach space, and obtained coincidence/fixed point theorems for such maps. On the other hand, Hadzic' [13] initiated the study of hybrid nonexpansive maps and its variants on convex metric spaces (convexity in the sense of Takahashi, see [13]) and convex probabilistic metric spaces.

COINCIDENCE AND FIXED POINT THEOREMS FOR HYBRID TYPE CONTRACTIONS

Several generalizations, variants and special cases of Theorem 1 have been obtained on several settings (see, for instance, [21], [24], [28], [34].

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[35], [37], [42], [43], [45], [48], [50] and [60-61]). Kaneko [21] proved that continuous and commuting maps $T:X \to CB(X)$ and $f:X \to X$ satisfying (1) and (1.1) have a coincidence in X provided X is complete. Generalizing this result Kaneko-Sessa [24] obtained the following :

Theorem 2

Let X be a complete metric space, and $f: X \to X$ and $T: X \to CB(X)$ be compatible continuous maps such that (1.1) and (4) hold. Then T and f have a coincidence.

We remark that the conditions of this theorem do not guarantee the existence of a common fixed point of T and f (see, for instance, Example 2 below). However, they [24, p. 260] (see also [21, Th. 1]) have shown that T and f with all the assumptions of Theorem 2 have a common fixed point whenever one of the following holds : either $fx \neq f^2x$ implies $fx \notin Tx$, or $fx \in Tx$ implies that $f^nx \to z$ for some $z \in X$. These conditions appear to be more acceptable than those cosmetized in [43].

In [50], Singh-Pant have obtained a coincidence theorem for two multivalued maps and a single-valued map in probabilistic analysis. However, the metric analogue of their result is a slightly improved by Theorem 3 (below), which is a special case of the main result of Singh-Ha-Cho [48].

Theorem 3

Let X be a metric space, $S,T:X \to CL(X)$ nd $f: X \to X$ such that $S(X) \cup T(X) \subset f(X)$, $f(X)$ is a complete subspace of X and

(6) $H(Sx, Ty) \leq k \max \{d(fx,fy), d(fx,Sx), d(fy,Ty), (1/2)[d(fx,Ty)+d(fy,Sx)]\}$

for some $0<k<1$ and all x,y in X. then S, T and f have a coincidence, i.e., there exists a point z in X such that $fz \in Sz \cap Tz$. Further, if for such a z, fz is a fixed point of f, then fz is also a fixed point of S (resp. T) provided f and S (resp. T) are weakly commuting at z, fz is a common fixed point of S and T provided f weakly commutes with each of S and T.

Indeed, in [48], the conclusions of Theorem 3 are obtained for conditions more general than those in Theorem 3. We remark that, as the following example shows, without the assumption "fz is a fixed point of f" (or without conditions leading to it) in Theorem 3, f, S and T need not have a common

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fixed point, even if the maps are continous and commuting on X. Moreover, "weakly" in Theorem 3 is superfluous, since, recall that, the weak commutativity at a coincidence point of T and f is equivalent to the commutativity at the coincidence point of T and f. Moreover, several results of [59] appear to be special cases of [48].

Example 2 [48]

Let $X = [0,1]$ and $Sx = Tx = \{0,1\}$, $fx = 1-x$ for all x in X. Then all the hypotheses of Theorem 3 are satisfied except that none of the coincidence values, viz., $f0$ or $f1$, is a fixed point of f. Note that f and S are continuous and commuting on X.

Coincidence and fixed point theorems for two multivalued and two single-valued maps on a metric space have been obtained among others by Kubiak [28], Naidu-Prasad [34], Naidu [61], Popa [37] and Sessa-Fisher [45]. Following Hadzic [12], Kubiak [28] established the following coincidence theorem.

Theorem 4 [28]

Let X be a complete metric space, f and g continuous maps from X into X, S and T closed maps from X into $CB(f(X) \cap g(X))$ such that $fSx = Sfx$, $gTx = Tgx$ for every x in X and

$$(7) \quad H(Sx, Ty) \leq k \max \{d(fx, gy), d(fx, Sx), d(gy, Ty), (1/2)[d(fx, Ty) + d(gy, Sx)]\}$$

for some $0<k<1$ and all x, y in X. Then there exists a point z in X such that $fz \in Sz$ and $gz \in Tz$.

Now, we have the following coincidence theorem for a sequence of multivalued maps, wherein N is the set of natural numbers.

Theorem 5

Let X be a complete metric space, f and g continuous maps from X into X and $\{Tn : n \in N\}$ a sequence of closed maps from X into $CL(f(X) \cap g(X))$ such that, for each n in N and each x is X, $fT_nx = T_nfx$, $gT_nx = T_ngx$, and

$$(8) \quad H(T_mx, T_nx) \leq$$

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$k \max \{d(fx,gy), d(fx,T_m x), d(gy,T_n y), 1/2[d(fx,T_n y)+d(gy,T_m x)]\}$ for some $0<k<1$ and all x,y in X, $m \neq n$. Then there exists a point z in X such that $fz \in T_n z$ and $gz \in T_n z$ for each $n \in N$.

Proof

An appropriate blend of the proofs of Kita [26] and Kubaik [28] will yield the result.

Note that Theorem 5 with $T_{2n-1} = S$ and $T_{2n} = T$ for each n in N slightly improves Theorem 4. Following Pal-Maiti [36] and the proof of Theorem 1 (above) (see [49]) we have the following :

Theorem 6

Let X be a metric space, $f: X \to X$ and $T : X \to CL(X)$ such that (1.1) obtains and $f(X)$ is a complete subspace of X. Further, let T and f satisfy, for x,y in X, at least one* of the following :

$H(fx,Tx)+H(fy,Ty) \leq ad(fx,fy),\ 1<a<2;$

$H(fx,Tx)+H(fy,Ty) \leq b\ [d(fx,Ty)+d(fy,Tx)+d(fx,fy)],\ 1/2<b<2/3;$

$H(fx,Tx)+H(fy,Ty)+H(Tx,Ty) \leq c\ [d(fx,Ty)+d(fy,Tx)],\ 1< c <3/2;$

and the condition (4) with $0<k<1$. Then there exists a point z in X such that $fz \in Tz$. Further, if fz is a fixed point of f, and if f and T commute at z, then fz is also a fixed point of T.

We remark that, in Theorem 6, if $f = id$ on X, then we get a result from [51], and if, in addition, T is a single-valued map on X, then we get an important fixed point theorem of Pal-Maiti [36].

As regards hybrid type contractions on a product space, a coincidence theorem has recently been obtained in [2] for two systems of multivalued maps and a system of single-valued maps on a finite product of metric spaces. It has been done by using a fixed matrix of non-negative numbers

*The intended meaning is that, for each pair of points x, y of X, one (or a set of more than one) of the four conditions is to be satisfied.

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(see [30]) and a new class of coordinatewise weakly commuting systems of single and multi-valued maps on the product space. It is too complicated to recall those results here. For an equivalent formulation of coordinatewise weakly commuting two sets of single-valued maps, refer [47].

HYBRID CONTRACTIVE MAPS

The following appears in [35].

Theorem 7

Let X be a metric space, $T : X \to CL(X)$ and $f: X \to X$ be such that $T(X) \subset f(X)$ and $f(X)$ is a compact subset of X. Suppose that for every $x, y \in X$,

(9) $H(Tx, Ty) < d(fx, fy)$, if $fx = fy$
$Tx = Ty$ if $fx = fy$

Then T and f have a coincidence.

For an attempt ;to obtain the conclusioin of this theorem under the strict inequality condition (4) with $k=1$, see [38]. Theorem 7 includes the following variant of Naimpally et al. [35, Cor. 3] (see also [11, p. 734]).

Theorem 8

Let X be a metric space and $f, g : X \to X$ be such that $g(X) \subset f(X)$, $f(X)$ is compact and

(10) $d(gx, gy) < d(fx, fy), fx \neq fy$

for x, y in X. Then f and g have a coincidence. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point.

Theorem 8 includes certain important fixed point theorems of Edelstein [9-10] and Jungck [18]. For more general results, refer [20]. If $X = R^n$ we have the following improved form of Theorem 8.

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Theorem 9 [46]

Let $f, g : R^n \to R^n$ be such that $g(R^n) \subset f(R^n)$, g is bounded and

$$|| gx - gy || < || fx - fy ||, fx \neq fy$$

for x, y in R^n. Then f and g have a coincidence. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point.

Theorem 9 with $f = id$ on R^n is an important result of Bridges et al. [5, Th. 2.2], since a contractive map of the closed unit ball in a Banach space need not have a fixed point (see [56] and also [53, p. 39]).

One may ask if Theorem 9 can be extended to the case when g is a multivalued map. In particular, we have the following :

Question : Let $T : R^n \to CL(R^n)$ and $f : R^n \to R^n$ such that $T(R^n) \subset f(R^n)$, $f(R^n)$ is bounded and (9) holds for $x, y \in R^n$. Will T and f have a coincidence ?

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Journal of Natural & Physical Sciences Vol. 5-8 (1991-94) pp 207-212

APPLICATION OF MODERN ASTRONOMY TO CERTAIN PROBLEMS OF HINDU ASTRONOMY♦

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(Received 03-12-1994)

ABSTRACT

Hindu methods of computing planetary positions of Sun, Moon etc. and calculating *Mandaphala* (Equation of Centre) and *Gatiphal* are some what lengthy. In this paper, we have suggested certain techniques of Modern Astronomy to compute planetary position, *Manadaphala* and *Gatiphala*. The methods are explained by taking examples from Hindu Astronomy.

Mathematics Subject Classification (1991) : 01A32

Keywords and Phrases : *Spars'a-Sthiti Khaṇḍa*, *Moksa-Sthiti-Khaṇḍa*; *Sammilana-Marda-Khaṇḍa*, *Unmilana-Marda-Khaṇḍa*.

INTRODUCTION

It appears that *Bhāskarācārya* II (b.1114 A.D.) had notions of Differential Calculus, but certainly not in the modern language. The above idea, 500 years before Newton and Leibntz, appears in his famous treatise *Siddhānta Śiromaṇi* (SS). For the exact computation of the daily motion of planets Bhāskara II introduced the concept of *Tatakālika Gati* (instantaneous velocity) by dividing the day into a large number of small intervals and compared the positions of planets at the end of successive intervals. If y and y′ are the mean anomalies of the planets at the end of consecutive intervals, then according to him

$$Hsin\ y' - Hsin\ y = (y' - y)\ Hcos\ y$$

(Here *Hsin* stands for Hindu-sine).
This result is equivalent to

$$\delta(sin\ y) = Cos\ y\ \delta y$$

(see, [3, p.110], [4, p. 146-148] and [1, p.92]).

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♦ Major part of this work was presented at the symposium on Science nad Technology in Ancient India, held at B.M. Birla Science Centre Hyderabad Dec. 1-2, 1990.

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Professor Srinivasiengar [op.cit.] observes that *Bhāskara* has gone deeper into Differential Calculus in *Siddhānta Śiromaṇi* and suggests that the differential coefficient vanishes at an extreme value of the function. In the case of lunar and solar eclipses, he says that celestial latitude (β) of the moon plays an important role for the determination of *Sparsa-Sthiti-Khaṇḍa* (the time between the moment of first contact and the middle of eclipse, i.e. half the duration of an eclipse), *Mokṣa-Sthiti-Khaṇḍa* (the time between the middle of the eclipse and the moment of the last contact), *Sammilana-Marda-Khaṇḍa* (the time between commencement of total eclipse and the middle of the eclipse) and *Unmilana-Marda-Khaṇḍa* (the time between the middle of the total eclipse and the end of the eclipse). It is well-known that celestial latitude (β) of the moon varies from moment to moment. For the exact computation of celestial latitude (β) at a point of the orbit of the moon, *Bhāskara* introduced a method in the following two verses of the SS :

स्थित्यर्धनाडीगुणिता स्वभुक्तिः
षष्ट्या ६० हता तद्रहितौ युतौ च ।
कृत्वेन्दुपातासन्नकृच्छराभ्यां
स्थित्यर्धमाद्यं स्फुटमन्तिमं च ॥
एवं विमदार्धफलोनयुक्त-
सपातचन्द्रोद्भवसायकाभ्याम् ।
पृथक् पृथक् पूर्ववदेवसिद्धे
स्फुट स्त आद्यन्त्यविमर्दखण्डे ॥

(The first verse gives the rectification of the times of *Sparsa-Sthiti-Khaṇḍa* and *Mokṣa-Sthiti-Khaṇḍa*, and the second verse gives the rectification of *Sammilana-Madra-Khaṇḍa* and *Unmilana-Marda-Khaṇḍa*).

However, when we follow the *Bhāskara's* method to calculate (β), we have to undergo somewhat lengthy calculations. Since the basic ideas of calculus are woven in the theory, we can apply modern calculus. For example to get the rectified value of β, let us differentiate the following with respect to β:

$Hsin\beta = (Hsin\lambda\ Hsin\ i)/R$

to get

$Hcos\beta\delta\beta = (Hsin\ i\ Hcos\lambda\delta\lambda)/R$

wherein i is the inclination of moon's orbit to ecliptic i.e., $i = 5°8'40''$, $R = 3438'$ and $(Hsin\ i)/R$ is a constant and λ = Moon's longitude - Node's longitude.

It may be mentioned that this formula gives in one stroke the rectified celestial latitude of the moon at the respectdive moments from which the

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respective rectified time may be computed immediately. The calculus may also be applied in several other situations.
We illustrate the method by an example.

Example : Compute the *Sparsa-Sthiti-Khaṇḍa* and *Mokṣa-Sthiti-Khanda* of partial lunar eclipse on August 6, 1990 at Hardwar.

Step 1

First we get the rectified planetary positions after making necessary corrections of *Desāntara, Carajyā, Bhujāntara* and *Udayāntara* of the sun, the moon and the lunar node with the help of *ahargaṇa.*

The following table gives the desired planetary position on August 6, 1990 at Hardwar.

Rectified longitude of the sun (Rls)	$3^R\ 20^o\ 25'\ 49''$
Rectified longitude of the moon (Rls)	$9^R\ 15^o\ 35'\ 19''$
Longitude of the lunar node	$9^R\ 12^o\ 59'\ 58''$
Difference of the sun and the lunar node	$282^o 59'58''-110^o\ 25'45''=172^o 34'13''$
Distance of the sun from the lunar node	$180^o-172^o\ 34'13''=7^o\ 25'47''<9^o$
Rlm-Rls	$9^R 15^o 35'19''-3^R 20^o 25'45''=5^R 25^o 9'34''$ $=175^o\ 9'\ 34''$
Difference to cover 180°	$180^o-175^o\ 9'\ 34''=4^o 50'\ 26''=17426''$
Rectified motion (*Gati*) of the sun (s)	$58'\ 18''\ 10'''$
Rectified motion (*Gati*) of the moon	$765'\ 1''$
m-s	$765'\ 1''-58'\ 18''\ 10''' = 42463''$
Duration of *Purnima* beginning from the sun rise i.e. from 0548 hrs (IST)	$17426 \times 60/42463$ Ghatis 24.62284812 Ghatis $24.62284812\times 24 \div 60$ hrs 9.849139246 hrs $9^h\ 50^m\ 56^s$

Evidently there is a lunar eclipse on Aug. 6, 1990.
Note that Rlm - Rls = 7° 25' 47" < 9°

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Step II:

Rectification of the sun, the moon and the lunar node at the end of *Purṇimā*

The following table shows the rectified longitudes of the sun, the moon and the lunar node at the end of *Purṇimā* (On Aug. 6, 1990).

Sun's motion during 60 *Ghatis*	$57'\ 18''\ 10''' = 57.302777'$
Sun's motion during *24.62284812 Ghatis*	$24.62284812 \times 57.302777/60 = 23'30''$
Rectified longitudeof the sun at the end of *Purṇimā*	$3^R\ 20^\circ\ 25'\ 45'' + 23'30'' = 3^R\ 20^\circ\ 49'\ 15''$
Moon's motion during *24.62284812 Ghatis*	$24.62284812 \times 765.01/60 = 5^\circ\ 13'56''$
Rectified longitude of the moon at the end of *Purṇimā*	$9^R\ 15^\circ\ 35' + 5^\circ 13'56'' = 9^R\ 20^\circ\ 49'\ 15''$
Mean daily motion of the lunar node	$0\text{-}3'\text{-}10''\text{-}48''\text{-}20''' = 3'\ 11'''$ approx.
Increment of the lunar node	$24.62284812 \times 3.180092592/60 = 1'18''$
Rectified longitude of the lunar node	$9^R\ 12^\circ\ 59'\ 58''\text{-}1'\ 18'' = 9^R\ 12^\circ\ 58'\ 40''$

ρ = Angular radius of the shadow cone	$= (2m/15\text{-}5s/12)'/2$ $= (2 \times 765.01/15\text{-}5 \times 57.3027/12)'/12$ $= 39.0626'$
r = Angular radius of the moon	$= 3m'/148 = 15.50'$
Moon's longitude with respect to the lunar node	$= 9^R\ 20^\circ\ 49'\ 15'' - 9^R\ 12^\circ\ 58'40''$ $= 7^\circ\ 50'\ 35''$
Celestial latitude (β) of the moon	$= (\text{Hsin}\ \lambda \times \text{Hsin}\ i)/3438$ $= 3438 \sin 7^\circ 50'35'' \times 270' \div 3438 = 36.38''$

Now $\rho + r - \beta = 17.93'$

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CONDITION FOR LUNAR ECLIPSE

Total lunar eclipse occurs when $\rho+r-\beta>2r$ and a partial lunar eclipse occurs when $\rho+r-\beta<2r$ since $\rho+r-\beta=17.39'<31'$. There is a partial lunar eclipse on Aug. 6, 1990.

Sparsā-Sthiti-Khaṇda $= [(\rho+r)^2-\beta^2]^{1/2}\times 60\div m\text{-}s$ *Ghatis*
(The first half duration of the eclipse) $= 3.427754943$ *Ghatis*

Therefore the length of the lunar eclipse is twice this period, i.e. *6.85550986 Ghatis*. This value is further refined 2 hrs. and 44.5 minutes.

Step III

The Following *Bhāskara II* (cf. second verse above), we get the following:

$$\lambda_1 = 7^\circ\ 6'\ 33' \quad \text{and} \quad \lambda_2 = 8^\circ\ 34'\ 27''$$

Wherein λ_1 (respectively λ_2) is the difference of the longitude of the moon and its node at the first (respectively last point of contact).

$$\text{Also } \beta_1 = 33.372' \quad \text{and} \quad \beta_2 = 39.90'$$

where β_1 and β_2 are the rectified values of β at the beginning and end of the eclipse. These vlaues may be calculated in a manner analogeous to β. (Further rectification of β_2 gives the same value).

STEP IV

Refined-*Sparsa-Sthiti-Khaṇda* $= [(\rho+r)^2-\beta_1^2]^{1/2}\times 60\div m\text{-}s$
$= 3.699716455$ *Ghatis*
$= 1$ *hr* 27 *m* 49 *sec.*

Mokṣa-Sthiti-Khaṇda $= \left[(\rho+r)^2-\beta_2^2\right]^{1/2}\times 60\div m-s$
$= 3.155232422$ *Ghatis*
$= 1$ *hr* 15 *m* 43 *sec.*

Now we apply modern calculus to get the rectified value of celestial latitude of the moon, we use the calculations of step I-III.Differentiating the formula :

$$Hsin\ \beta = 270'\times Hsin\ \lambda \div 3438\ Kal\bar{a}$$

we get

$$\delta\beta = 270'\times \cos\lambda\delta\lambda \div 3438\cos\beta\ Kal\bar{a}$$

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$= 270' \cos 7^\circ 6' 53'' \times 1^\circ 27' 44'' \div 3438 \cos 36.36'$ *Kalā*

$= 6.48$ *Kalā*

In fact this is the approximate average change in the celestial latitude of the moon.

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फार्म - ४

प्राकृतिक एवं भौतिकीय विज्ञानशोध पत्रिका

सम्मिलित खण्ड ५-८ १९९१-१९९४

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